THESIS TITLE:

BLACK BOX STATE ESTIMATION

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Abstract

This research work was motivated by the work of Bardyn et al [3] which they proposed Bell's inequalities as a tool to estimate the quality of a black box "entangled pair" source. In the paper, under the restriction of pure states, they derived a lower bound for the Mayers & Yao fidelity based on the observed CHSH violation. We conducted numerical studies in $\mathbb{C}^4 \otimes \mathbb{C}^4$ to investigate Mayers & Yao fidelity and trace distance and compared the results with the analytical bounds given by [3]. It was found that the numerical pure states saturates the two qubits analytical bound instead of the qudits bound. In addition, we showed an example where the closest ideal states can be infinitely many. The later part of the research was shifted to black box teleportation, certain black box teleportation schemes were proposed and how to distinguish between black box teleportation and genuine quantum teleportation.

Keywords:

Entanglement, Quantum State Estimation, Bell's Inequalities, Quantum Teleportation

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Chapter 1

Introduction

1.1 Quantum physics and state estimation

The development of quantum mechanics since the early twentieth century has been fast and vast. The very first revolution of quantum physics comes from the property of wave-particle duality that concluded with inventions like transistors and lasers. However this wave-particle duality is a single particle interference. With two quantum particles, entanglement appears. Einstein, Podolsky and Rosen [1] in 1935 pointed out that quantum mechanics predicts strong correlations between measurements on two space-like separated systems. However the attempt to interpret quantum correlations in the view of a realist demands that quantum mechanics must be complemented with an additional classical variables that are hidden from the observers. This in short implies *local realism*, which reads as results of measurements on a localized system are fully deterministic and cannot be influenced by any other distant event. Later in 1964, John Bell [2] came with a mathematical theorem called *Bell theorem* that defines a limit on correlations for any local realism models. Then it seems that one should either drop *locality* or *realism* or even drop both! It is intuitively correct to say that any physical entity always has some information that exist regardless of measurements, e.g., a sealed container that contains a basketball will always contain the basketball regardless whether one opens it or not. However, experiments have shown that if one uses quantum entities (photons, electrons, etc) for the experiments, then statistics from the experiments show that Bell theorem is violated by quantum physics. If we assume *locality* then we have to accept that the measurement outcomes do not exists before the measurement, it lacks intuition but it is a fact that was tested and verified by many scientific groups around the world. It is also well known that these non-local correlations can be achieved from incompatible measurements on entangled particles that are distributed among space-like separated parties.

However, the fundamental relationship between entanglement and non-local correlations is indeed not that obvious. It was initially thought that entanglement and non-local nature of any quantum state goes in the same direction and they are very much identical to each other as quantum information processing resources. But it was shown that for some Bell inequalities, the quantum state that achieve the maximal violation is not a maximally entangled state. Nevertheless, one can still invoke Bell's inequalities to detect for non-local quantum states and these non-local quantum states imply entanglement. It is also interesting to ask if some non-local correlations are observed, (e.g., experiment statistics violates Bell's inequality) can one learn something about the possible state of the entangled system? Well, this sounds like a problem of quantum state estimation already.

In principle, quantum state estimation is a method that provides the complete description of a system, or equivalently to achieve the possible maximal knowledge of the state. It is clear that in classical regime, one can always fully recover the characteristics of an unknown classical object as one can always copy many copies of the original system. However in quantum physics, there is a fundamental limit on our knowledge of a quantum state. That is, if one is only given a single copy of an unknown quantum system then one cannot learn the complete description of it. This principle is given by the *quantum no-cloning theorem.* The demonstration of quantum no-cloning theorem requires only the linearity of quantum dynamics: Suppose one has an unknown quantum state in the following two levels system representation where the basis is given as $\{|0\rangle, |1\rangle\}$

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1.1}$$

There exists a quantum cloning machine such that it will produce an exact copy of the unknown state:

$$|0\rangle \otimes |\mathrm{R}\rangle \rightarrow |0\rangle \otimes |0\rangle \tag{1.2}$$

$$|1\rangle \otimes |\mathrm{R}\rangle \rightarrow |1\rangle \otimes |1\rangle \tag{1.3}$$

now let the unknown state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ be processed with such a cloning machine

$$|\phi\rangle \otimes |\mathrm{R}\rangle \rightarrow \alpha|00\rangle + \beta|11\rangle \tag{1.4}$$

and this obviously is not a exact clone of the initial unknown state! Because if cloning is possible, then the end state should be

$$|\phi\rangle \otimes |\phi\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle \tag{1.5}$$

We have demonstrated that quantum no-cloning theorem doesn't allows one to clone an unknown quantum state and it is evident that quantum state estimation demands many copies of identical states. In addition, it seems that to characterize an unknown quantum state requires the prior knowledge of its physical properties. For example, photon does not interact with magnetic fields hence one cannot learn more about a photon with a Stern-Gerlach apparatus (okay, you learn that the unknown particle has no charge.). So that is to say, If one does not have the prior knowledge of physical properties(spins, polarization, etc) of the quantum object, then it seems that it is impossible to perform state tomography. However in quantum systems, there is a subtle property that allows composite quantum systems to have more as a whole than just a summation of its subsystems. It is also precisely this exclusive nature of quantum systems that one can use it to estimate the state of a black box that promise to generate entangled pairs.

1.2 Objective of honours research

The objective of the honours research is to develop new mathematical tools or approaches that allows one to estimate an entangled black box source via Bell inequalities. The primary aim for the honours is to extend the work of Bardyn et al [3] *Device independent state estimation based on Bell's inequalities* to arbitrary bipartite quantum states. However, it is also the aim of the project to conduct numerical studies to gain some intuitions about solutions. In addition to this work, we also explored the possibility of using teleportation dependency on entanglement to estimate the bound on the average fidelity that differentiate between quantum black boxes and classical black boxes.

1.3 Organization of thesis

The organization of the thesis is as follows:

1. Chapter 2: The preliminaries aims to introduce to unfamiliar readers the basic tools and techniques required to follow the research work in the thesis. In particular, the distance measures and CHSH inequality are the main constructs of the research work.

2. Chapter 3: §(3.1) of this chapter introduces the work of Bardyn et al [3], figure of merits and formulation of problem. §3.2 are ordered as such

   (a) The proposed family of ideal states that can give $2\sqrt{2}$ CHSH violation for $\mathbb{C}^4 \otimes \mathbb{C}^4$.

   (b) The extension of Horodecki et al necessary and sufficient condition for $\mathbb{C}^2 \otimes \mathbb{C}^2$ to certain higher even dimension systems. The results were compared with the generalized CHSH operator.

   (c) The numerical studies of Mayers & Yao fidelity and trace distance. The results were compared with the theoretical results by [3].

   (d) An counter example of state that has infinitely many ideal states was presented and implications are discussed.

3. Chapter 4: This chapter takes a different direction in the approach of black box state estimation and instead of using Bell's inequalities we use teleportation scheme to distinguish between black box classical and quantum teleportation schemes.
    (a) Two models of classical teleportation scheme were adapted to black box scenarios.
    (b) Proposed a protocol to distinguish black box teleportation and genuine quantum teleportation.
4. Chapter 5: The discussions and results are highlighted and discussed in depth. Future work details the approach on how to conduct quantum apparatus estimation.

Chapter 2

Preliminaries

2.1 Quantum states and density matrix

In quantum physics, any quantum system can be described by a representative space called the Hilbert space(generally infinite dimension) and the Hilbert space is a complex vector space with Hermitian inner product by mathematical framework. The dimension of the representation space is then fully determined by the degree of freedom of the specified property possessed by the physical system. For example, for a spin $\frac{1}{2}$ particle the internal degree of freedom (spin) can be described by a $\mathbb{C}^2$ Hilbert space. Here one can ascribe the spin states as kets $\{|0\rangle, |1\rangle\}$ in the Dirac notation and these states are perfectly distinguishable and correspond to the spin outcomes $\{+\frac{1}{2}, -\frac{1}{2}\}$. Any pure state of a two-level system can be written as $|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$ where $\alpha, \beta \in \mathbb{C}$ characterize the quantum state in the spin state basis. In general, quantum systems are microscopic systems such that one can only use indirect observation to obtain information about it. In precise definition, the state of knowledge(of the specified property) of the quantum state is fully characterized by the state vector residing inside the Hilbert space. This in turn implies that the Quantum-Hilbert space formalism is simply a mathematical platform for one to express the state of knowledge with respect to the eigenstates of the measurement apparatus. Often the observer's knowledge of the system is incomplete even if one carefully prepares the quantum system. For example instead of the intended preparation of all spin $+\frac{1}{2}$ states, there may exists some mixture of spin $-\frac{1}{2}$ states.

It is then obvious that representation in terms of kets can only be used to describe superposition of states and not statistical mixtures of pure states.

Density operators

In order to represent any statistical mixtures of pure states, we introduce the definition of density operator of a pure state $|\psi\rangle$ as

$$\rho := |\psi\rangle\langle\psi| \tag{2.1}$$

This is also called the density matrix of a pure quantum states and it has the follow properties:

1. It is positive and thus implies Hermitian, $\rho \geq 0$
2. The trace of the density matrix is always equal to unity, $\mathrm{tr}(\rho) = 1$ and this implies normalization.

It is easy to see that given any density matrix, one can always diagonalize the matrix in some basis such that the matrix can be (spectral) decomposed into its eigenvectors

$$\rho = \sum_x \mathrm{P}_X(x)|\phi_x\rangle\langle\phi_x| \tag{2.2}$$

where the eigenvalues $\mathrm{P}_X(x)$ is the probability of getting pure state $|\phi_x\rangle$. The density operator assumes pure state if and only if one of the eigenvalues of ρ is 1, $\mathrm{tr}(\rho^2) = 1$. In the language of density operators, a distinct advantage is that one can also describe a classical system that in the Hilbert space. We define these classical states as

$$\rho_{\text{classical}} = \sum_x \mathrm{P}_X(x)\sigma_x^{\text{classical}} \tag{2.3}$$

whereby $\sigma_x^{\text{classical}}$ represents the x state of the classical system, e.g., head or tail of the coin. In general these classical states are perfectly distinguishable (orthogonal in Hilbert Space), which later one can see that quantum theory of information is actually a mathematical generalization of the classical information theory.

2.1.1 Dynamics of (isolated) quantum states

Now that we have defined quantum states and their properties, one may ask what can we do with these quantum states? As discussed above, a quantum state is nothing more than a mathematical representation of our perceived knowledge of the physical system one is interested in. Strictly speaking, one is only using quantum physics to predict measurable properties of the system and not to fully describe the system. These described systems are not directly observed (naked eye but see [4]) but are measured through machines So it leaves only two possible paths:

1. Quantum Dynamics: Time evolution of quantum systems
2. Quantum Measurement: To obtain information about properties of the described system, e.g., particle's spin, photon's polarization, etc

So in principle, how does one describe the evolution of a quantum system? The answer can be clued from classical physics, Liouville dynamics. The classical Liouville equation is a linear operation that describes the time evolution of phase space distribution function from time t_i to t_f and is defined as

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{d} \left(\frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial \rho}{\partial p_i} \right) = 0 \tag{2.4}$$

$$\frac{\partial \rho}{\partial t} = - \sum_{i=1}^{d} \left(\frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial \rho}{\partial p_i} \right) = 0 \tag{2.5}$$

where the RHS of eqn. (2.5) is the Poisson bracket and is closely related to the commutator in Quantum mechanics and $H(q, p, t)$ is the classical Hamiltonian.

Koopman's theorem

The Koopman's theorem states that the phase space density distribution does not change under Lioville dynamics, i.e., given two phase space distributions, the fidelity or inner product of these two distributions is invariant under Liouville dynamics:

$$\frac{d\langle \rho(q, p, t) | \sigma(q, p, t) \rangle}{dt} = \frac{d}{dt} \int \rho(q, p, t) \sigma(q, p, t) dq dp = 0 \tag{2.6}$$

Isolated Quantum evolution

In quantum physics, the time development of quantum states is given by the Schrodinger's equation for time evolution operator [5]

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = HU(t,t_0) \tag{2.7}$$

solving the equation for time independent Hamiltonian yields

$$U(t,t_0) = \exp\left(-\frac{i}{\hbar}H(t-t_0)\right) \tag{2.8}$$

so in fact, the unitary $U(t,t_0)$ operator encodes the full information of the evolution and is deterministic and reversible, i.e., given any initial condition $|\psi, t_0\rangle$

$$U(t,t_0\rangle|\psi, t_0\rangle = |\psi, t\rangle \tag{2.9}$$

$$U^\dagger(t,t_0)U(t,t_0) = \mathbb{I} \tag{2.10}$$

Hence the inner product of the quantum state under the quantum evolution is conserved similarly to the classical anagloue.

$$\langle\psi, t_0|\psi, t_0\rangle = \langle\psi, t|\psi, t\rangle = \langle\psi, t_0|U^\dagger(t,t_0)U(t,t_0)|\psi, t_0\rangle = 1 \tag{2.11}$$

2.1.2 Measurement on quantum systems

Consider a quantum state that was prepared by some process in the past and given as ρ in the present. A generalized operation is an operation that takes the present state ρ to some other future state ρ'.

$$\Lambda : \rho \to \rho' \tag{2.12}$$

and this operation in general can have m distinguishable outcomes given as λ_m and each outcomes has a corresponding density operator or quantum state ρ_m. As defined in

Quantum formalism, the probability of random variable $X = \lambda_m$ outcome is defined as:

$$\mathrm{P}_{X|\rho}(\lambda_m) := \mathrm{tr}(\rho\Pi_m) \tag{2.13}$$

where usually the elements Π_m are usually called positive operator valued measure (POVMs) but I will denote them here as generalized observables. Naturally, the probability of an event is always real and ranges between $0 \leq \mathrm{P}_X \leq 1$. The first property of a probability implies that the operators Π_m must be Hermitian. Another constraint on the generalized observables is the normalization of the probabilities, $\sum_m \mathrm{P}_{X|\rho}(\lambda_m) = 1$ implies that the summation of all operators must be equal to identity, $\mathbb{I}$.

$$\sum_m \mathrm{P}_{X|\rho}(\lambda_m) = \sum_m \mathrm{tr}(\rho\Pi_m) = \mathrm{tr}(\rho\mathbb{I}) = 1 \tag{2.14}$$

These two conditions are the only conditions for any generalized observables that takes a quantum state to another quantum state with the corresponding probability. However in the situation of quantum state discrimination where one tries to distinguish between quantum states, only the probability distributions matters in the evaluation. This will be discussed in detail in the section of trace distance measure. Now suppose that one can write the generalized observables in the form of

$$A_m = U_m \Pi_m^{\frac{1}{2}} \tag{2.15}$$

where U_m is any unitary operator and hence satisfy reversibility $U_m^\dagger U_m = \mathbb{I}$, hence the generalized observables can be written as

$$\Pi_m = A_m^\dagger A_m \tag{2.16}$$

and eqn. (2.13) now reads

$$\mathrm{P}_{X|\rho}(\lambda_m) := \mathrm{tr}(A_m \rho A_m^\dagger) \tag{2.17}$$

and the post-measurement normalized quantum state that corresponds to outcome λ_m is

$$\rho'_{\lambda_m} = \frac{A_m \rho A_m^\dagger}{\mathrm{tr}(A_m \rho A_m^\dagger)} \tag{2.18}$$

2.2 Composite systems and tensor product structure

So far, we have only discussed about single quantum system state description and generalized measurements. But in reality, there is a more bizarre quantum phenomena called *entanglement* that doesn't has a direct classical analogue (but see [6]) to it. The description of composite quantum systems requires the treatment of tensor product structures and later we will show that *non-signaling* can be derived from it.

2.2.1 Subsystems

The product Hilbert space $\mathcal{H}_{A\otimes B}$ of two subsystems $\mathcal{H}_A$, $\mathcal{H}_B$ (in general, the dimensions of the Hilbert spaces need no be the same) is defined to be the tensor product of the two subsystems (Hilbert spaces)

$$\mathcal{H}_{A\otimes B} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{2.19}$$

Also for each vectors $|\phi_A\rangle \in \mathcal{H}_A$, $|\phi_B\rangle \in \mathcal{H}_B$, the product vector $|\phi_{A\otimes B}\rangle \in \mathcal{H}_{A\otimes B}$ can be written as

$$|\phi_{A\otimes B}\rangle := |\phi_A\rangle \otimes |\phi_B\rangle \tag{2.20}$$

Now, considering bipartite systems ($\mathcal{H}_A^n \otimes \mathcal{H}_B^m$), one can make use of the Schmidt decomposition defined as

$$|\phi_{A\otimes B}\rangle := \sum_{i=1}^{k} \sqrt{\lambda^i} |u_A^i\rangle \otimes |v_B^i\rangle \tag{2.21}$$

Where $k \leq \min(n, m)$ and $|u_A^i\rangle, |v_A^i\rangle$ are bi-orthonormal eigenvectors of reduced density operator $\rho_A = \mathrm{tr}(\rho_{A\otimes B})$ and $\rho_B = \mathrm{tr}(\rho_{A\otimes B})$ respectively.

2.2.2 Product Operators

Let O_A be a linear operator defined on the Hilbert Space of system A $\mathcal{H}_A$ and Q_B be a linear operator on $\mathcal{H}_B$, then the product of the operators is defined as

$$R_{A\otimes B} := O_A \otimes Q_B \tag{2.22}$$

This definition of bi-local operator acts only locally on the subsystems and given a product state

$$R_{A\otimes B}|\phi_{A\otimes B}\rangle := (O_A \otimes Q_B)(|\phi_A\rangle \otimes |\phi_B\rangle) = O_A|\phi_A\rangle \otimes Q_B|\phi_B\rangle \tag{2.23}$$

and the bi-local operator can be equivalently written as

$$R_{A\otimes B} = (O_A \otimes \mathbb{I}_B)(\mathbb{I}_A \otimes Q_B) \tag{2.24}$$

One point to emphasis is that given a bi-local operator and a product quantum state, one can always factorize the joint expectation of the bi-local operator into expectation into local expectation values.

$$\langle\phi_{A\otimes B}|R_{A\otimes B}|\phi_{A\otimes B}\rangle = \langle\phi_A|O_A|\phi_A\rangle\langle\phi_B|Q_B|\phi_B\rangle \tag{2.25}$$

However, if the given quantum state is not a product state (Schmidt number > 1) then eqn (2.25) is no longer valid.

2.2.3 Non-signaling

A conditional probability distribution $\mathrm{P}_{XY|ab}(x, y)$ is defined to be non-signaling if for all inputs a, b

$$\mathrm{P}_{X|ab} = \mathrm{P}_{X|a} \tag{2.26}$$

$$\mathrm{P}_{Y|ab} = \mathrm{P}_{Y|b} \tag{2.27}$$

where $P_{X|ab}$ and $P_{Y|ab}$ are marginals of $P_{XY|ab}$. The interpretation behind this condition is that the choice of input at Alice(Bob)'s laboratories cannot influence the outcomes of Bob(Alice)'s measurements. This is essentially the mathematical statement which states that no information can be communicated instantaneously. In quantum mechanics framework, let Alice and Bob's measurements by represented by generalized measurements $\{\Pi_x^a\}_x$ and $\{\Gamma_y^b\}_y$ where a, b are the possible inputs. These generalized measurements satisfy normalization, $\sum_x \Pi_x^a = \mathbb{I}_A$ and $\sum_y \Gamma_y^b = \mathbb{I}_B$. Given ρ_{AB}, the joint probability of getting outcomes x, y that is conditioned on the inputs a, b is

$$P_{XY|ab,\rho_{AB}}(x, y) = \mathrm{tr}(\Pi_x^a \otimes \Gamma_y^b \rho_{AB}) \tag{2.28}$$

the marginal of X reads

$$\begin{aligned} \sum_y P_{XY|ab,\rho_{AB}}(x, y) &= P_{X|ab,\rho_{AB}} = \sum_y \mathrm{tr}(\Pi_x^a \otimes \Gamma_y^b \rho_{AB}) \\ P_{X|ab,\rho_{AB}} &= \mathrm{tr}(\Pi_x^a \otimes \mathbb{I}_B \rho_{AB}) = P_{X|a,\rho_{AB}} \end{aligned} \tag{2.29}$$

where the probability distribution of Alice is independent of Bob's choice of inputs which satisfy eqn. (2.27). This also holds for the marginal of Y.

2.3 States distinguishability measures

This topic of state distinguishability measures is a little misleading but is befitting to the objective of differentiating quantum states under the context of experiments. It is misleading in a sense that the only meaningful information one can get from the experiments are the frequencies of each outcomes and total number of outcomes and not the abstract density operators. With these information, one can then define the probability of observing any particular outcome within the measurement context. Hence to distinguish quantum states one needs to perform some measurements and compare the probability distributions for the given outcomes. The simplest case of distinguishing two probability distributions that are not identical (if the distributions are identical then the probability of guessing it correctly will be just $\frac{1}{2}$ and this isn't that interesting in our formulation) will be the starting point of our discussion. In which, we introduce the idea of distinguishability via error probability and show that the classical Kolmogorov trace distance is closely related to it. In the following subsections will be the introduction of the quantum trace distance measure and we will show that the quantum trace distance and fidelity measures are the generalization of the classical distance measures.

2.3.1 Distinguishability and Errors

In order to discuss the notion of distinguishability of probability distributions, one can consider a simple classical probability game [7]. The game is defined as such:

Guess the correct distribution game

Given two probability distributions, $\mathrm{P}_X(x)$ and $\mathrm{Q}_X(x)$ where $x = 1, ..., n$, a player samples blindly once from either $\mathrm{P}_X(x)$ or $\mathrm{Q}_X(x)$ without knowing the identities of the distributions. If the player guesses the correct distribution then he wins, if not he loses the game. In order to make the game more attractive, the player is allowed to know the probability of the distribution which he is sampling, σ_{P} and $\sigma_{\mathrm{Q}} = 1 - \sigma_{\mathrm{P}}$ respectively. It is obvious that if the two probability distributions are distinct then the player may gain some meaningful information about the identity of the sampled distribution. In such a game, the player has only the information of probability of getting either distributions prior to the sample and may try to say something about the probability of error in his single guess.

Probability of error

First lets define the guess/decision function as

$$\delta : \{1, ..., n\} \to \{0, 1\} \tag{2.30}$$

that is the method of guessing the the correct distribution. So the probability of guessing it wrongly will be

$$\mathrm{P}_{\mathrm{error}}(\delta) = \sigma_{\mathrm{P}} \mathrm{P}(\delta = 1|0) + \sigma_{\mathrm{Q}} \mathrm{P}(\delta = 0|1) \tag{2.31}$$

where $\mathrm{P}(\delta = 0|1)$ is the probability that the guess/decision is $\mathrm{P}_X(x)$ but the correct distribution is $\mathrm{Q}_X(x)$ and the converse is given as $\mathrm{P}(\delta = 1|0)$. It can be shown [7] that the probability of error is

$$\mathrm{P}_{\mathrm{error}}(\delta) = \sum_{x=1}^{n} (\mathrm{P}_X(x) + \mathrm{Q}_X(x))(1 - \max(\mathrm{P}(0|x), \mathrm{P}(1|x)) \tag{2.32}$$

$$\mathrm{P}_{\mathrm{error}}(\delta) = \sum_{x=1}^{n} \min(\sigma_{\mathrm{P}} \mathrm{P}_X(x), \sigma_{\mathrm{Q}} \mathrm{Q}_X(x)) \tag{2.33}$$

2.3.2 Distinguishing probability distributions: Kolmogorov measure

Given that the probability of error in choosing the distributions is given by eqn. (2.33), one can let $\sigma_{\mathrm{Q}}, \sigma_{\mathrm{P}} = 1/2$ which is the most pessimistic situation for the player. The probability of error now reads

$$\mathrm{P}_{\mathrm{error}}(\delta) = \frac{1}{2} \sum_{x=1}^{n} \min(\mathrm{P}_X(x), \mathrm{Q}_X(x)) \tag{2.34}$$

on another hand, one can write the probability of success as $\mathrm{P}_{\mathrm{Success}} = 1 - \mathrm{P}_{\mathrm{error}}$ and obtains

$$\mathrm{P}_{\mathrm{Success}} = 1 - \frac{1}{2} \sum_{x=1}^{n} \min(\mathrm{P}_X(x), \mathrm{Q}_X(x)) \tag{2.35}$$

which can be rewritten as

$$\mathrm{P}_{\mathrm{Success}} = \frac{1}{2} + \frac{1}{2} \left[1 - \sum_{x=1}^{n} \min(\mathrm{P}_X(x), \mathrm{Q}_X(x)) \right] \tag{2.36}$$

The probability of distinguishability of two probability distributions is given by the Kolmogorov distance which gives the geometric interpretation of distance between two probability distributions.

$$D(\mathrm{P}(x), \mathrm{Q}(x)) = \frac{1}{2} \sum_{x=1}^{n} |\mathrm{P}(x) - \mathrm{Q}(x)| \tag{2.37}$$

or equivalently given by the R.H.S square bracket of eqn. (2.36)

$$D(\mathrm{P}(x), \mathrm{Q}(x)) = 1 - \sum_{x=1}^{n} \min(\mathrm{P}(x), \mathrm{Q}(x)) \tag{2.38}$$

The trace distance then satisfy the following properties

1. It is symmetric and $0 \leq D(\mathrm{P}(x), \mathrm{Q}(x)) \leq 1$

2. Positive and is zero if and only if $\mathrm{P}(x) = \mathrm{Q}(x)$

3. It satisfy the triangle inequality, $D(\mathrm{P}(x), \mathrm{Q}(x)) \leq D(\mathrm{P}(x), \mathrm{R}(x)) + D(\mathrm{R}(x), \mathrm{Q}(x))$

4. $D(\mathrm{P}(x), \mathrm{Q}(x)) = 1$ if and only if $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ have distinct support.

Another important property of the trace distance is that it can only decrease under the operation of taking the marginals.

$$\begin{aligned} D(\mathrm{P}_{XY}(x,y), \mathrm{Q}_{XY}(x,y)) &= \frac{1}{2}\sum_{x,y} |\mathrm{P}(x,y) - \mathrm{Q}(x,y)| \\ \frac{1}{2}\sum_{x,y} |\mathrm{P}(x,y) - \mathrm{Q}(x,y)| &\geq \frac{1}{2}\sum_{x} |\sum_{y} \mathrm{P}(x,y) - \mathrm{Q}(x,y)| \\ &= \frac{1}{2}\sum_{x} |\mathrm{P}(x) - \mathrm{Q}(x)| \end{aligned} \tag{2.39}$$

2.3.3 Quantum generalization of Kolomogov measures

The main mathematical tool behind the black box state estimation work is the use of trace distance between quantum states. That is given two quantum states $\rho, \sigma \in \mathcal{H}$, the trace distance between the two quantum states is defined as

$$D_{\mathrm{QM}}(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1 \tag{2.40}$$

where the trace norm of an operator A is defined as

$$\|A\|_1 := \mathrm{tr}\sqrt{A^* A} \tag{2.41}$$

the quantum trace distance eqn. (2.40) carries the same properties as the classical Kolmogorov trace distance and is essentially a metric defined on the space of the density operators. Also, it has the property that it is invariant under unitary operations that is similar to the fidelity measures for classical phase space distributions eqn (2.6). Now one can notice that given two density operators $\rho_{\mathrm{classical}}, \sigma_{\mathrm{classical}}$ (cf. eqn. (2.3)) that are characterized by only classical distributions $\mathrm{P}_X(x)$ and $\mathrm{Q}_X(x)$ respectively, the quantum trace distance reduces to the classical trace distance.

$$
\begin{aligned}
D_{\mathrm{QM}}(\rho_{\text{classical}}, \sigma_{\text{classical}}) &= \frac{1}{2}\|\rho_{\text{classical}} - \sigma_{\text{classical}}\|_1 \\
\frac{1}{2}\|\rho_{\text{classical}} - \sigma_{\text{classical}}\|_1 &= \frac{1}{2}\|\sum_x \mathrm{P}_X(x)\mathbb{I}_{\text{classical}} - \sum_x \mathrm{Q}_X(x)\mathbb{I}_{\text{classical}}\|_1 \\
&= \frac{1}{2}\sum_x |\mathrm{P}_X(x) - \mathrm{Q}_X(x)| \qquad (2.42)
\end{aligned}
$$

one also must be careful to note that quantum trace distance depends not only on the density operators ρ, σ but also on the generalized measurements (k inputs and m outcomes) $\sum_{x=1}^m \Pi_x^k = \mathbb{I}$ involved in the evaluation, i.e.,

$$\mathrm{P}_{X|\Pi_x^k,\rho}(x) = \mathrm{tr}(\rho\Pi_x^k) \qquad (2.43)$$

$$\mathrm{Q}_{X|\Pi_x^k,\sigma}(x) = \mathrm{tr}(\sigma\Pi_x^k) \qquad (2.44)$$

The evalution of eqn. (2.40) will be then to maximize the classical trace distance between $\mathrm{P}_{X|\Pi_x^k,\rho}(x)$ and $\mathrm{Q}_{X|\Pi_x^k,\sigma}(x)$ over all possible generalized measurements.

$$\frac{1}{2}\|\rho - \sigma\|_1 = \max_{\Pi} \frac{1}{2}\sum_x |P_{X|\Pi_x^k,\rho}(x) - Q_{X|\Pi_x^k,\sigma}(x)| \qquad (2.45)$$

Therefore the quantum trace distance between two quantum states can be interpreted as the maximum distinguishing probability, i.e., the maximum probability that one can detect a difference between ρ and σ. Hence given two quantum states, one can obtain an upper bound on the probability that these two states can be distinguished and this clearly has a strong operational definition, i.e., the state distinguishability game. Suppose that Alice prepares two quantum states and distributes them statistically, $\frac{1}{2}\rho + \frac{1}{2}\sigma$. Now Bob has to make a guess with some generalized measurements to distinguish between the two quantum states. As we have shown in §2.3.2, Bob's probability of success is then $\frac{1}{2} + \frac{1}{2}D_{\mathrm{QM}}$. So if the quantum states are perfectly distinguishable then Bob can win the game with probability of success equal 1. However if the quantum states are identical in the measurement context, his probability of success is $\frac{1}{2}$. Furthermore, trace distance has a geometrical interpretation as well as it is a metric defined on the operator space and is symmetric in terms of density operators. This geometrical picture is obvious if one considers two-level systems (qubits) that has a corresponding $\mathbb{R}^3$ representation via the Bloch Sphere.

2.3.4 Fidelity as an alternative distinguishability measure

An alternative to trace distance is another useful distance measure called fidelity measure. The fidelity measure has a vague operational definition as compared to the trace distance eqn. (2.40) but is often easier to compute. The quantum fidelity measure is very similar to the classical fidelity (inner product) and reads

$$F_{\mathrm{QM}}(\rho,\sigma) := \|\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\|_1^2 \tag{2.46}$$

Now if one of the states is a pure state,e.g., $\sigma = |\psi\rangle\langle\psi|$ then fidelity reduces to the expectation value of ρ given state σ

$$F_{\mathrm{QM}}(\rho,\sigma) = \mathrm{tr}(\sqrt{\sigma^{\frac{1}{2}}\rho\sigma^{\frac{1}{2}}})^2 = \sqrt{\langle\psi|\rho|\psi\rangle}^2 = \langle\psi|\rho|\psi\rangle \tag{2.47}$$

and if both quantum states are pure,

$$F_{\mathrm{QM}}(\rho,\sigma) = \langle\psi|\phi\rangle\langle\phi|\psi\rangle = |\langle\psi|\phi\rangle|^2 \tag{2.48}$$

Thus the fidelity between two pure quantum states is simply the probability of transition between $|\psi\rangle$ and $|\phi\rangle$. As one can see, the fidelity is unity if and only if the quantum states are identical and is zero if the quantum states are on orthogonal subspaces.

The quantum trace distance eqn. (2.40) and quantum fidelity measure has a direct relation given as

$$D_{\mathrm{QM}}(\rho,\sigma) \leq \sqrt{1 - F_{\mathrm{QM}}(\rho,\sigma)} \tag{2.49}$$

Where the equality is satisfied for only pure quantum states.

2.4 Quantum correlations

As mentioned in the introduction, classical physics allows one to predict any measurement outcomes with unity probability if and only if all influences (forces, fields, etc) and specifications of measurements are taken into account. However, within the framework of quantum formalization one can only make probabilistic predictions of any observations. That is the uncertainty in one's measurement of a quantum system is intrinsic and fundamental as oppose to the uncertainty of classical observations. This fundamental uncertainty is called the Heisenberg's uncertainty relation and it is the catalysis that lifted determinism from our quantum experiments. It is also this relationship between non-commuting physical properties or incompatible measurements that give rise to Einstein, Podolsky and Rosen's 1935 famous paper [1] that questioned the completeness of quantum mechanics. The basic argument is that any individual (local measurements) outcomes may depend on an additional parameter that is famously called the *classical local hidden variables* or in modern language *shared randomness*. Theories involving shared randomness are generally deterministic and probabilities exist because of the users lack of knowledge (which is the hidden variables). Now, if one believes that no meaningful information can be communicated faster than speed of light (*non-signaling*), then there are no consistent ways to predict experimental results by using local hidden variables models.

2.4.1 Bell's inequalities

John Bell in 1964 [2] derived a theorem (an inequality) that can be interpreted as: "Physical theories compatible with special relativity and outcomes of measurements are fundamentally deterministic satisfy the inequality". In short, if one believes in local realism then his/her prediction of experiment's results should not violate the Bell's inequality. However if one is to do the computation under the framework of quantum physics, then one can achieve violation of Bell's inequalities. In 1982, Alain Aspect et al [8] conducted the CHSH experiment and showed that the statistics from the experiments violated the CHSH Bell's inequality by five standard deviations, this crucial result proved that Bell's inequalities are violated by quantum objects (Photons source in [8]). So it was shown that in nature, there exists such a channel that allows non-local correlations that cannot be simulated by local hidden variables under space-like separated conditions. Now consider the possible motivations behind performing or constructing a Bell's inequality experiment:

1. The motivation post 1964: Bell introduced the Bell theorem and gave an inequality that allows one to apply it to any physical theories. This theorem under quantum mechanics predicted stronger correlation than local realism models. Experimental demonstrations of Bell's inequality were needed to verify the predictions of quantum physics. This motivation leaded to experiments like [8].

2. Motivation post 1982: Even as Aspect et al produced the first verification that quantum prediction is correct, there were still loop-holes. There are still ongoing groups who are trying to close all these loop-holes simultaneously.

3. Motivation post 1991: Ekert [9] proposed to use the degree of violation of the CHSH inequality as a bound on the security of quantum key distribution. This result motivated the community to develop experimental quantum key distribution systems like [10] and thus the need for setting up CHSH experiments.

4. Motivation of 21st Century: Device independent systems: The introduction of Mayers & Yao [11] and Bardyn et al [3] works on self-testing quantum devices gave the new direction that one can use Bell's inequality setups to estimate the quality of quantum systems.

In the first two motivations, the aim is to study and learn more about the non-locality property of quantum physics while for the last two motivations are to use the non-local property of quantum systems to bound certain parameters of the systems that we have no prior knowledge of. It is precisely the latter that defines the scope of this honours project and the work of Bardyn et al will be introduced in Chapter 3.

Bipartite Bell's inequalities

In general, Bell's inequalities are constraints on some functions of correlators of measurement's results and they are satisfied by all separable states i.e., $\rho = \sum_i P_i \rho_i^A \otimes \rho_i^B$ and $\sum_i P_i = 1$. Here, the definition of a bipartite bell's inequality [12] is given as

$$B(c) \geq \sum_{i=1}^{n} \sum_{j=1}^{m} c(i,j) E(i,j) \tag{2.50}$$

$c(i,j)$ are some real coefficients, $E(i,j)$ is the correlation function of the product of the observables and $B(c)$ is the maximal possible value of R.H.S of eqn. (2.50). As all Bell's inequalities are satisfied by shared randomness, one can without loss of generality define the correlation functions as

$$E(i,j) = \int d\lambda \rho(\lambda) A(i,\lambda) B(j,\lambda) \tag{2.51}$$

where $A(i,\lambda)$, $B(j,\lambda)$ are deterministic and local functions (alternative notation: local response functions) that given λ and i,j always output a specific result. So in the correlation functions, what is not known or rather hidden is just the local deterministic strategy distribution $\rho(\lambda)$ and this is what give rise to probabilities in LHV theories.

2.4.2 Clauser, Horne, Shimony and Holt inequality

Since Bell introduction of Bell's inequality, there has been many versions and variations of inequalities that are adapted to n_A, n_B settings and m_A, m_B outcomes each settings. For $n_A = n_B = 2$ and $m_A = m_b = 2$, there is only one tight inequality [13]: Clauser, Horne, Shimony and Holt inequality [14].

$$-2 \leq E_{XY|\vec{a},\vec{b}} + E_{XY|\vec{a},\vec{b}'} + E_{XY|\vec{a}',\vec{b}} - E_{XY|\vec{a}',\vec{b}'} \leq 2 \tag{2.52}$$

where each measurement device has binary input and binary output, $\{\vec{a}, \vec{a}'\}$ for Alice and $\{\vec{b}, \vec{b}'\}$ for Bob. The correlator reads

$$E_{XY|\vec{a},\vec{b}} = P_{XY|\vec{a},\vec{b}}(+,+) + P_{XY|\vec{a},\vec{b}}(-,-) - P_{XY|\vec{a},\vec{b}}(-,+) - P_{XY|\vec{a},\vec{b}}(+,-) \tag{2.53}$$

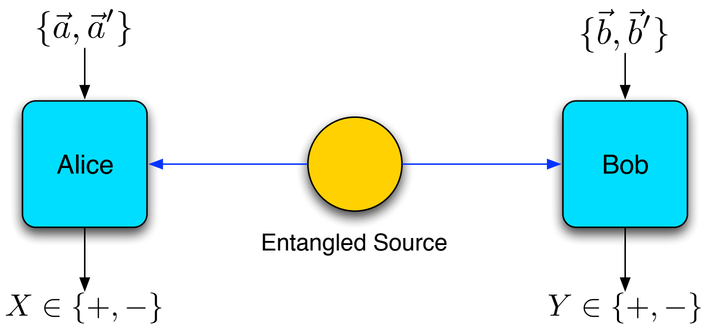

Figure 2.1: An abstract illustration of a simple black box computer. The user is given the specfication of the black box computer and a black box device that supposely perform accordingly to the specification. In this illustration, a user has to enter binary values to n inputs and will locally receive a real valued outcome.

Local bound of CHSH and Cirel'son's bound

Using a pre-established agreement (LHV), one can describe the eqn. (2.52) with eqn. (2.51) and rewrite the CHSH as

$$\int d\lambda \rho(\lambda)[A(\vec{a}, \lambda)(B(\vec{b}, \lambda) + B(\vec{b}', \lambda)) + A(\vec{a}', \lambda)(B(\vec{b}, \lambda) - B(\vec{b}', \lambda))] \tag{2.54}$$

and $A(i, \lambda), B(j, \lambda) = \pm 1$ implies that the CHSH inequality for any specific λ is bounded by

$$-2 \le [A(\vec{a}, \lambda)(B(\vec{b}, \lambda) + B(\vec{b}', \lambda)) + A(\vec{a}', \lambda)(B(\vec{b}, \lambda) - B(\vec{b}', \lambda))] \le 2 \tag{2.55}$$

which is exactly eqn. (2.52). Now if one uses quantum states or explicitly the Bell states,

$$|\Phi^{\pm}\rangle := \frac{1}{\sqrt{2}}(|e_0\rangle \otimes |f_0\rangle \pm |e_1\rangle \otimes |f_1\rangle) \tag{2.56}$$

$$|\Psi^{\pm}\rangle := \frac{1}{\sqrt{2}}(|e_0\rangle \otimes |f_1\rangle \pm |e_1\rangle \otimes |f_0\rangle) \tag{2.57}$$

where $\{|e_0\rangle, |e_1\rangle\} \in \mathcal{H}_A$ and $\{|f_0\rangle, |f_1\rangle\} \in \mathcal{H}_b$ are bi-orthonormal states. Next define the quantum measurement operators as Hermitian operators

$$A(\vec{n}) = \vec{n} \cdot \vec{\sigma} \tag{2.58}$$

with ± 1 eigenvalues and $\vec{n} \in \mathbb{R}^3$, and one can get the following relation with the local identity operations

$$
\begin{aligned}
A(\vec{a})^2 &= A(\vec{a}')^2 = \mathbb{I}_A \\
B(\vec{b})^2 &= B(\vec{b}')^2 = \mathbb{I}_B
\end{aligned}
\tag{2.59}
$$

Next the Bell operator is defined [15] as

$$
\mathcal{B} = A(\vec{a}) \otimes [B(\vec{b}) + B(\vec{b}')] + A(\vec{a}') \otimes [B(\vec{b}) - B(\vec{b}')] \tag{2.60}
$$

$$
\mathcal{B}^2 = 4\mathbb{I}_A \otimes \mathbb{I}_B - [A(\vec{a}), A(\vec{a}')][B(\vec{b}), B(\vec{b}')] \tag{2.61}
$$

here, one can take the *sup norm* of eqn (2.48) and with $\|A + B\|_{\text{sup}} \leq \|A\|_{\text{sup}} + \|B\|_{\text{sup}}$, one gets

$$
\begin{aligned}
\|\mathcal{B}^2\|_{\text{sup}} &= \|4\mathbb{I}_A \otimes \mathbb{I}_B - [A(\vec{a}), A(\vec{a}')][B(\vec{b}), B(\vec{b}')]\|_{\text{sup}} \\
&\leq \|4\mathbb{I}_A \otimes \mathbb{I}_B\|_{\text{sup}} + \|[A(\vec{a}), A(\vec{a}')][B(\vec{b}), B(\vec{b}')]\|_{\text{sup}} \\
&= 4 + 4\|[A(\vec{a})\|_{\text{sup}} \cdot \|[A(\vec{a})'\|_{\text{sup}} \cdot \|[B(\vec{b})\|_{\text{sup}} \cdot \|[B(\vec{b})'\|_{\text{sup}} \\
&= 8
\end{aligned}
\tag{2.62}
$$

as the Bell operator $\mathcal{B}$ is a Hermitian operator, i.e., $A^\dagger A = AA = A^2$

$$
\|\mathcal{B}^2\|_{\text{sup}} = \|\mathcal{B}\|^2_{\text{sup}} \leq 8 \tag{2.63}
$$

and since the *sup norm* of $\mathcal{B}$ is the maximum eigenvalue of $\sqrt{\mathcal{B}^\dagger \mathcal{B}} = \mathcal{B}$, one can conclude that the maximum eigenvalue of $\mathcal{B}$ is bounded by $2\sqrt{2}$, which is the Cirel'son's bound [16] for the CHSH inequality. This bound is valid for any arbitrary dimension systems as long as the quantum measurements satisfy eqn. (2.59), i.e., the observables have eigenvalues ± 1.

Optimization of CHSH inequality

Now that we have introduced the CHSH inequality, one in general can interpret it as a function that takes a density operators $\rho_{AB} \in \mathcal{H}_{A\otimes B}$ to $\mathbb{R}$

$$S_{\mathrm{CHSH}} : \rho_{AB} \to s \in [-2\sqrt{2}, 2\sqrt{2}] \tag{2.64}$$

where in principle the mapping is given by $S_{\mathrm{CHSH}}(\rho_{AB}) := \mathrm{tr}(\mathcal{B}\rho_{AB}) = s$ and is dependent on following parameters

1. The choice of measurement settings, $\{\vec{a}, \vec{a}', \vec{b}, \vec{b}'\}$,
2. The choice of measurement bases, e.g., $\sigma_3 = |e_0\rangle\langle e_0| - |e_1\rangle\langle e_1|$
3. The *entanglement factor* of ρ_{AB}, e.g., singlet fraction, negativity, etc .

Now one can notice that even if the CHSH experiment involves a maximally entangled states (Bell states), this doesn't guarantee that the mapping brings it to the maximal violation of CHSH inequality, $\pm 2\sqrt{2}$. It means that in order to achieve maximal violation, one can either fix the quantum state ρ_{AB} to find the optimal measurement settings or one can fix the settings and find the optimal quantum states. The latter is less interesting as we are now more interested in estimating how non-local a quantum state is under the context of CHSH inequality. In the following demonstration, one can invoke the Horodecki et al [17] necessary and sufficient condition for violation of CHSH inequality for a given arbitrary $\mathbb{C}^2 \otimes \mathbb{C}^2$ quantum composite system.

Necessary and sufficient condition for violating CHSH inequality by arbitrary two qubits

Given two qubits that have interacted in the past and are now deposited or transmitted to Alice and Bob laboratories, one can describe the composite state as

$$\begin{aligned} \rho_{AB} &= \frac{1}{2}\left(\mathbb{I}_A + \vec{n}\cdot\vec{\sigma}_A\right) \otimes \frac{1}{2}\left(\mathbb{I}_B + \vec{m}\cdot\vec{\sigma}_B\right) \\ &= \frac{1}{4}\left(\mathbb{I}_A \otimes \mathbb{I}_B + \vec{n}\cdot\vec{\sigma}_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \vec{m}\cdot\vec{\sigma}_B + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j\right) \end{aligned} \tag{2.65}$$

where $t_{ij} = \mathrm{tr}(\rho_{AB}\sigma_i \otimes \sigma_j)$. Now lets define a matrix $U(\rho_{AB}) := T^T T$ which is symmetric and hence is diagonal in some bases. It was shown by [17] that the information that is encoded into the T real matrix is related to the maximal violation of the CHSH inequality:

$$\max_{\mathcal{B}} |S(\rho_{AB})_{\mathrm{CHSH}}| = 2\sqrt{M(\rho_{AB})} \tag{2.66}$$

where $M(\rho_{AB}) := \lambda_1 + \lambda_2$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are the ordered positive eigenvalues of matrix $U(\rho_{AB})$. In fact, the T matrix encodes the quantum correlations given by the quantum state ρ_{AB} and its explicit form is given as

$$T := 2\begin{pmatrix} \mathrm{Re}(a\vec{d} + b\vec{c}) & \mathrm{Im}(a\vec{d} - b\vec{c}) & \mathrm{Re}(a\vec{c} + b\vec{d}) \\ \mathrm{Im}(a\vec{d} + b\vec{c}) & \mathrm{Re}(-a\vec{d} + b\vec{c}) & \mathrm{Im}(a\vec{c} - b\vec{d}) \\ \mathrm{Re}(a\vec{b} - c\vec{d}) & \mathrm{Im}(a\vec{b} - c\vec{d}) & \frac{1}{2}(|a|^2 + |d|^2 - |b|^2 - |c|^2) \end{pmatrix} \tag{2.67}$$

where $|\phi\rangle = a|e_0, f_0\rangle + b|e_0, f_1\rangle + c|e_1, f_0\rangle + d|e_1, f_1\rangle$ and $\{a, b, c, d\} \in \mathbb{C}$. Now as an example, we consider the Werner state [18]

$$\rho_{\mathrm{W}} = \mathrm{P}|\Phi^+\rangle\langle\Phi^+| + (1 - \mathrm{P})\frac{\mathbb{I}}{4} \tag{2.68}$$

As the Bell operator is traceless, i.e., $\mathrm{tr}(\mathcal{B}) = 0$, the necessary and sufficient condition for the Werner state to be non-local is $\mathrm{P} > \frac{1}{\sqrt{2}}$. Next, consider a mixture of local states and a maximally entangled state,

$$\rho = \mathrm{P}|\Phi^+\rangle\langle\Phi^+| + (1 - \mathrm{P})|e_0, f_0\rangle\langle e_0, f_0| \tag{2.69}$$

The T matrix is then the convex combination of $T(|\Phi^+\rangle)$ and $T(|e_0, f_0\rangle)$

$$T(|\Phi^+\rangle) = \mathrm{diag}(+, -, +) \tag{2.70}$$

$$T(|e_0, f_0\rangle) = \mathrm{diag}(0, 0, +) \tag{2.71}$$

$$T(\rho) = \mathrm{P}\mathrm{diag}(+, -, +) + (1 - \mathrm{P})\mathrm{diag}(0, 0, +) \tag{2.72}$$

where now $M(\rho) = 1 + \mathrm{P}^2$ implies the maximal violation of the CHSH inequality with eqn. (2.66) is $2\sqrt{1+\mathrm{P}^2}$.

2.4.3 Entanglement as a communication resource

As we have seen in the earlier section, we used Bell's inequalities as a mathematical measure to estimate the amount of non-locality for some given quantum states. However, there also exist other types of non-local measures that detect the non-locality of states. One famous example is the Hardy non-locality test [19]. Now one can ask a very relevant question,"is entanglement the same as non-locality in terms of resource?" As an example, one performs an CHSH experiment in space-like separated laboratories and achieve a violation of $\simeq 2.422$ for ρ_1 and $\simeq 2.622$ for ρ_2, can one say that ρ_2 is much more entangled than ρ_1? In general, this is true only for two qubits systems and not true for anything more complicated than that [20]. In [20], the authors gave an example in the next simplest case of two qutrits: using the Collins, Gisin, Linden, Massar and Popescu (CGLMP) inequality for a maximally entangled qutrits pair, they obtained a violation of 2.873. However, they pointed out that Acin et al [21] found a higher violation with a non-maximally entangled state (with the same settings as the maximally entangled states, but this settings and non-maximally entangled state were checked to be optimal). This implies that non-locality and entanglement are in general different resources. Nevertheless, entanglement [22] has been essential in achieving communication feats that classical communication cannot achieve, to list some:

1. Quantum teleportation
2. Quantum dense coding
3. Entanglement swapping
4. Quantum Key Distributions

Here, I will introduce the concept of quantum teleportation only as it is one of the on-going work on black box state estimation.

2.4.4 Quantum Teleporation with qubits

For ages, people have been fascinated by the possibility of teleportation of beings from one place to another space-like separated location instantaneously, this fascination is evident as block buster movies like Star-Trek use "teleportation" frequently. Now teleportation is possible in nature as predicted by quantum mechanics and it respects special relativity, i.e., the teleportation cannot be completed faster than speed of light. If that is the case, what is so fantastic about quantum teleportation if it is still bounded by speed of light? Well, it transmits partial information of the "to be teleported" particle via a *channel of nature* that is well described by quantum mechanics. Unfortunately, for every single use of the *channel* requires us to sacrifice a well distributed entangled pair. That is, we need to pay to use this channel. Also, the above statement sort of trivializes the hassle in performing such an experimental, because one still needs to have perfect quantum apparatus like the Bell state measurement, unitary operations, minimum distribution losses, etc. Nevertheless, it is still a wonderful application of quantum physics! so lets review the standard quantum teleportation protocol which was discovered by Bennett, Brassard, Crepeau, Jozsa, Peres and Wootters in 1993 [23]:

Quantum teleportation protocol

Objective: Alice wants to "teleport" an unknown state to Bob.

1. Alice receives an unknown quantum state $|\phi(?)\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$

2. Alice does a joint measurement (Bell state measurement) on both $|\phi(?)\rangle$ and her qubit of the entangled pair $|\Phi^+\rangle$. This measurement is essentially a generalized measurement with four outcomes $\{00, 01, 10, 11\}$.

3. Via the classical channel (e.g., telephone, GSM, etc) she informs Bob about the outcome of her Bell state measurement.

4. Bob then does the correct unitary rotation that corresponds to the outcome obtained by Alice. This of course requires that Alice and Bob has some pre-established agreement that will give the correct interpretation of the outcomes of the Bell state

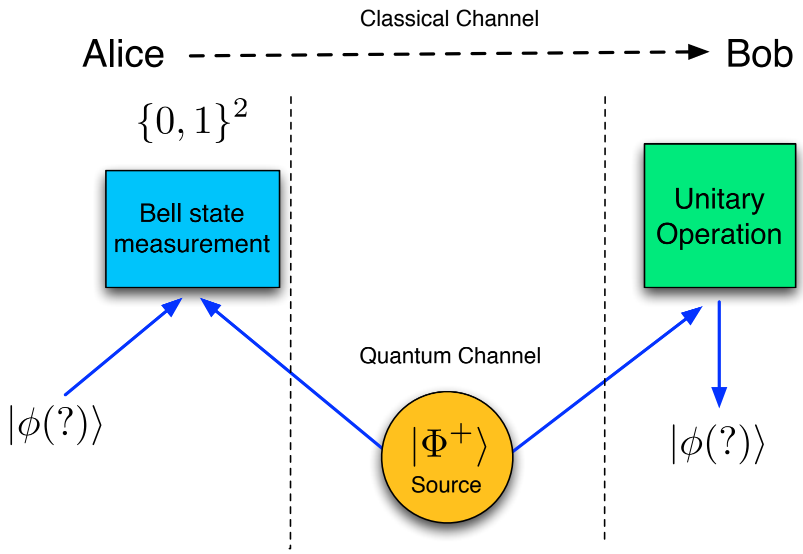

Figure 2.2: A schematic of the teleportation protocol: Alice receives an unknown quantum state $|\phi(?)\rangle$ and perform a Bell state measurement on both $|\phi(?)\rangle$ and her qubit (Entangled pair). The outcome is then communicated to Bob via the classical channel and Bob does the corresponding unitary operations, e.g., σ_Z corresponds to 00 from Alice.

measurement.

5. After the unitary operation, Bob will then exactly receives $|\phi(?)\rangle$.

To be able to achieve faithful teleportation depends on a few crucial components. They are mainly perfect quantum apparatus, source distributes maximally entangled states and users have knowledge of the entangled source. One can view the teleportation as a well-coordinated perfect transmission of 2 real numbers, i.e., α and β with just two classical bits. This is indeed very amazing! because if one wishes to represent any two real numbers, in principle will require infinite amount of classical bits, for example try to represent π in terms of bits. Another astonishing feat that is a direct consequence of teleportation is the Entanglement swapping protocol [24], whereby after the execution of the swapping protocol, two quantum objects that *never* interacted can be entangled!

Comments on preliminaries

In writing the preliminaries, I hope that unfamiliar readers can follow the later chapters in which I introduce the main idea of black box (device independent) state estimation by Bardyn et al and the original research studies that I have conducted. In addition to this preliminaries, it is highly recommended that unfamiliar readers also explore the rich amount of literatures of quantum information science.

Chapter 3

State estimation with Bell inequalities:CHSH

3.1 Description of problem and composite black box systems

In the previous section, we see that entanglement is a property of quantum systems that has no direct classical analog (however see [25] where the authors defined a interesting classical substitute for entanglement correlation called secret correlation.). Also, it is essentially an information processing resource that allows one to perform communication tasks like QKD, teleportation, dense coding, etc that classical communication technologies cannot achieve. Now imagine in the situation whereby the human race has advanced towards quantum technologies saturation such that every current technologies now involved quantum physics (classical computer involves quantum physics as well; electrical transistors. In particular a single black box cannot tell you much about the quantum physics as we seen in the introduction chapter. However the situation changes if one considers composite black boxes systems. Because in classical physics, composite systems are nothing more than summation of individual systems dynamics and their mutual interactions, while quantum composite systems as we have seen can have entanglement between subsystems! This is totally absence in classical physics! because in classical physics if one gains complete knowledge about the subsystems then one can fully describe the system as a whole, however with entangled systems the information is only accessible if the composite system is considered. It is precisely this property that makes composite black box physics as the next approach to distinguish between specification and statistics of the black boxes.

In this chapter, the work was mainly motivated by the recent publication by Bardyn, Liew, Massar, McKague and Scarani [3] where they proposed Bell's inequalities as an approach to estimate the quality of a black box source that is specified to produce entangled pairs. The observed violation of CHSH, S_{obs} is then used to bound the trace distance between the ideal entangled source and the observed black box source. However, the authors did not managed to obtain a general bound for arbitrary bipartite states(for arbitrary pure states, they did) but suggested a few possible figure of merits that may be meaningful in the black box context. In §(3.1), I introduce the work of Bardyn et al, §(3.2) the numerical studies for arbitrary two four levels system $\mathbb{C}^4 \otimes \mathbb{C}^4$ that I have done.

Formulation of problem

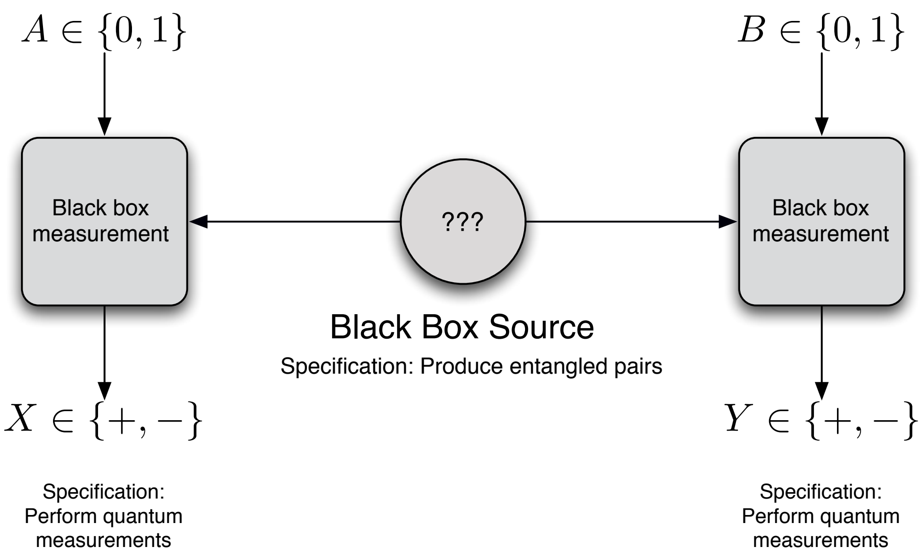

Figure 3.1: An abstract illustration of composite black box system:Three black boxes are defined in this figure, (1) black box source which is specified to produce entangled pairs (2) Alice's black box that is specifed to perform projective measurement on her side of the entangled pair (3) Bob's black box that has the same specification as Alice's black box. .

In preliminaries §§(2.4), one of the way to check for quantum correlations is the Bell's inequalities. In such a measure, there is a dependency on measurement settings, bases and amount of entanglement of the source. Now consider the composite black box setup in fig. (3.1), where there are three black boxes, the source and two measurement boxes. I define the assumptions of black box physics as:

1. The dynamics of a black box are completely unknown to the users
2. A black box is assumed to be isolated and hence there is no interaction with the environment but under the composite black box system, the black boxes may have some form of non-signalling communications between black boxes that are unknown to the users.
3. The physics inside the black box can be described by the physics outside the black box.
4. The availability of *free will* to the users is defined as the black box boundary.

In this setup, the amount of information that Alice and Bob (now the users) can obtain are only the choice of measurements $\{0, 1\}$ and $\mathrm{P}_{X|A}(\pm)$, $\mathrm{P}_{X|B}(\pm)$ and $\mathrm{P}_{XY|AB}(\pm, \pm)$. With these information, they are suppose to estimate the quality of the source, .i.e., to quantitatively distinguish the observed black box source and the specifications.
The main goal of this section is to formulate a quantitative measure between the ideal states and observed black box source based on the CHSH violation observed. Here, ideal states is defined as states that belongs to a set $\mathcal{S}$ of states that can achieve absolute maximum violation of the CHSH inequality

$$\left\{\rho_{\text{ideal}} \in \mathcal{S} | S_{\text{CHSH}} : \rho_{\text{ideal}} \mapsto \operatorname{tr}\left(\mathcal{B}\rho_{\text{ideal}}\right) \leq 2\sqrt{2}\right\} \tag{3.1}$$

For pure states that achieve the above condition are fully characterized [26, 27] and it is in the form

$$|\Phi\rangle = \sum_i c_i \,|\Psi_i\rangle \tag{3.2}$$

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}}\left(|2i-1,\, 2i-1\rangle + |2i,\, 2i\rangle\right) \tag{3.3}$$

where $|\Psi_i\rangle$ are two qubit maximally entangled state in $\mathbb{C}^2 \otimes \mathbb{C}^2$ subspace. Also, in general the source can also produce mixed states due to imperfection of the black box or losses between black boxes. One can then invoke the results from Mayers and Yao [11]. It states that if the source and measuring apparatus are imperfect, then the real system is essentially identical to the ideal specifications up to a local change of basis on each party. Therefore the most general state from the set $\mathcal{S}$ will be of the form

$$\rho_{\text{ideal}} = U_{\text{A}} \otimes U_{\text{B}} \left(\Phi^+ \otimes \sigma\right) U_{\text{A}}^\dagger \otimes U_{\text{B}}^\dagger \tag{3.4}$$

and Φ^+ is defined as $\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

3.1.1 Figure of merits

Given an abitrary state from the black box, one can define the geometric distance between it and the closest ideal state ρ_{ideal} by the trace distance eqn. (2.40)

$$\delta_{\text{MY}}\left(\rho\right) := \min_{\rho_{\text{ideal}}} \frac{1}{2} \left\|\rho - \rho_{\text{ideal}}\right\|_1 \tag{3.5}$$

where the Mayer and Yao trace distance is minimized over the set $\mathcal{S}$. This gives the maximal distinguishing probability that the actual source will differ from the ideal source as shown in [3], the authors set to find an upper bound to the eqn. (3.5) as a function of the observed CHSH operation,

$$\delta_{\text{MY}}\left(\rho\right) \leq \mathcal{D}_{\text{MY}}\left(S_{\text{obs}}\right) \tag{3.6}$$

This upper bound can then be computed by optimizing the CHSH operation with some optimal measurement settings

$$\delta_{\text{MY}}\left(\rho\right) \leq \max_{\rho : S_{\max}(\rho) \geq S_{\text{obs}}} \left\{ \min_{\rho_{\text{ideal}}} \frac{1}{2} \left\|\rho - \rho_{\text{ideal}}\right\|_1 \right\} \tag{3.7}$$

Alternatively, the problem can be formulated in terms of fidelity and one can then maximize the fidelity over the set $\mathcal{S}$

$$F_{\mathrm{MY}}(\rho) := \max_{\rho_{\mathrm{ideal}}} \left(\left\| \rho^{\frac{1}{2}} \rho_{\mathrm{ideal}}^{\frac{1}{2}} \right\|_1 \right)^2 \tag{3.8}$$

$$F_{\mathrm{MY}}(\rho) \geq \mathcal{F}_{\mathrm{MY}}(S_{\mathrm{obs}}) \tag{3.9}$$

where the problem is now formulated equivalently in terms of fidelity.

3.1.2 Under the assumption of two qubits

In principle, how does one go ahead and compute the R.H.S of eqn (3.7)? As Bardyn et al shown in [3], one can gain some intuitions by restricting the black box source to two qubits space and then try to extend the solutions to higher dimensions. In their example of two qubits black box source, they used the spectral decomposition [28] of the CHSH Bell operator eqn. (2.60):

$$\mathcal{B} := \sum_{k=1}^{4} \lambda_k |\Phi_k\rangle\langle\Phi_k| \tag{3.10}$$

where the eigenvalues of the CHSH Bell operator are $\{\lambda_1.\lambda_2, \lambda_3 = -\lambda_2, \lambda_4 = -\lambda_1\}$ and the eigenvalues satisfy the following condition

$$\lambda_1^2 + \lambda_2^2 = 8 \tag{3.11}$$

and $\{|\Phi_k\rangle\}$ are the Bell states eqn. (2.57) or eigenkets of the Bell operators. Let the black box source be ρ and with eqn. (3.10)

$$\mathrm{tr}(\mathcal{B}\rho) = \sum_{k=1}^{4} \lambda_k \langle\Phi_k|\rho|\Phi_k\rangle \tag{3.12}$$

let $\lambda_1, \lambda_2 \geq 0$ and now in order to maximize $S(\rho)$, let $|\Phi_1\rangle$ satisfy the condition such that $|\Phi_1\rangle$ is the closest Bell state to ρ. This implies that

$$S_{\max}(\rho) \leq \lambda_1 \langle\Phi_1|\rho|\Phi_1\rangle + \lambda_2 \langle\Phi_2|\rho|\Phi_2\rangle \tag{3.13}$$

Now one can recognize that $\langle\Phi_1|\rho|\Phi_1\rangle = F_{\mathrm{MY}}(\rho)$

$$S_{\max}(\rho) \leq \lambda_1 F_{\mathrm{MY}}(\rho) + \lambda_2(1 - F_{\mathrm{MY}}(\rho)) \tag{3.14}$$

here one can maximize the Bell operator's settings by using the following inequality:

$$A\cos(\theta) + B\sin(\theta) \leq \sqrt{A^2 + B^2} \tag{3.15}$$

from condition of eqn.(3.11), one can rewrite the $\lambda_1 = 2\sqrt{2}\cos(\theta)$ and $\lambda_2 = 2\sqrt{2}\sin(\theta)$ and this implies that one can perform the following maximization

$$S_{\max}(\rho) \leq \lambda_1 F_{\mathrm{MY}}(\rho) + \lambda_2(1 - F_{\mathrm{MY}}(\rho)) \leq 2\sqrt{2}\sqrt{F_{\mathrm{MY}}(\rho)^2 + (1 - F_{\mathrm{MY}}(\rho))^2 - 1} \tag{3.16}$$

$$2F_{\mathrm{MY}}(\rho)^2 - 2F_{\mathrm{MY}}(\rho) + (1 - \frac{S_{\max}(\rho)}{4}^2) \geq 0 \tag{3.17}$$

solving for $F_{\mathrm{MY}}(\rho)$ yields

$$F_{\mathrm{MY}}(\rho) \geq \frac{1}{2}\left(1 + \sqrt{\frac{S_{\mathrm{obs}}}{4}^2 - 1}\right) \tag{3.18}$$

this equation gives the lower bound to the Mayers & Yao Fidelity eqn. (3.9) in terms of the observed violation of CHSH computed from the experiment statistics. Now if the black box source is restricted to just two qubits subspaces then the problem has been solved by Bardyn et al, however in higher dimensions the quantum states that saturate the Tirel'son bound are not just the Bell states eqn. (2.57) but states of the form eqn. (3.2). It was also shown by Braunstein et al [29] that any mixtures of eigenkets of the Bell operator for arbitrary dimensions can give maximal violation. Now before we move on to arbitrary dimension black box source, we will introduce a paper by Gisin and Peres [30] whose conjecture was that if a pure state is in their Schmidt basis and their phases are ordered and real positive, the optimal CHSH observables are given as block diagonals.

3.1.3 Gisin-Peres's CHSH pure states conjecture

Let a bipartite pure states be

$$|\Phi\rangle_{AB} = \sum_{i=1}^{d} c_i |e_i\rangle_A \otimes |f_i\rangle_B \tag{3.19}$$

where $c_1 \geq c_2 \geq c_3 \geq \cdots \geq c_d \geq 0 \in \mathbb{R}$, and define Γ_z and Γ_x as block diagonal matrices, i.e.,

$$\Gamma_z := \oplus_{i=1}^{d/2} \sigma_z \tag{3.20}$$

$$\Gamma_x := \oplus_{i=1}^{d/2} \sigma_x \tag{3.21}$$

with Alice and Bob's observables with eigenvalues of ± 1 defined as

$$A(\alpha) := \Gamma_z \mathrm{cos}(\alpha) + \Gamma_x \mathrm{sin}(\alpha) + \Pi \tag{3.22}$$

$$B(\beta) := \Gamma_z \mathrm{cos}(\beta) + \Gamma_x \mathrm{sin}(\beta) + \Pi \tag{3.23}$$

where the matrix Π is defined as $[\Pi]_{dd} = (d \bmod 2)$. Now if one uses these observables for the CHSH operator eqn. (2.60),

$$\mathcal{B} = A(\alpha) \otimes [B(\beta) + B(\beta')] + A(\alpha') \otimes [B(\beta) - B(\beta')] \tag{3.24}$$

and set $\alpha = 0, \alpha' = \pi/2$ and $\beta = -\beta'$, one obtains the optimal CHSH bound:

$$\langle \Phi | \mathcal{B} | \Phi \rangle_{AB} \leq 2\sqrt{(1-\gamma)^2 + 4(c_1 c_2 + c_3 c_4 + \cdots)^2} + 2\gamma \tag{3.25}$$

where γ is given as $c_d^2(d \bmod 2)$. Now as one can notice for $d = 2$, the measurement scheme eqn. (3.23) is consistent with results obtain by Horodecki et al [17] eqn. (2.66) and Popescu & Rorhlich [31]. Unfortunately this measurement scheme is yet to be analytically proven to be the optimal for states of eqn. (3.19), however numerical results by Liang & Doherty [32] suggested that for arbitrary pure bipartite states, the optimal measurement scheme of CHSH operator was always given by eqn. (3.23). Using the CHSH bound eqn. (3.25) but block-wise, Bardyn et al showed that if one relaxes the restriction of two qubits to two qudits pure states, the Mayers & Yao's fidelity is lower bounded by

$$F_{\text{MY}}\left(\rho\right) \geq \frac{1}{4(\sqrt{2}-1)}[S_{\text{obs}}+2\sqrt{2}-4] \tag{3.26}$$

Which in principle, if one could show that the $F_{\text{MY}}\left(\rho\right)$ for any arbitrary bipartite quantum state is always lower bounded by the $F_{\text{MY}}^{\text{pure}}\left(\rho\right)$ for pure state, then the problem is essentially solved, i.e., the user just employ the bound for bipartite pure states for any observed CHSH violation S_{obs} and use that bound to estimate the quality of the black box.

3.1.4 Definition of idea states, $\mathcal{S}$

In §4.1, the definition of ideal states reads

$$\left\{\rho_{\text{ideal}} \in \mathcal{S} | S_{\text{CHSH}} : \rho_{\text{ideal}} \mapsto \text{tr}\left(\mathcal{B}\rho_{\text{ideal}}\right) \leq 2\sqrt{2}\right\} \tag{3.27}$$

In $\mathbb{C}^2 \otimes \mathbb{C}^2$ Hilbert space, only the Bell states satisfy the above definition and are unique up to local unitaries, i.e, one can always transform one of the Bell basis to another Bell basis via local unitaries, e.g., $\mathbb{I}_A \otimes U_B$. So a natural question one can ask is: "can any mixture of the Bell states still saturate the Cirel'son bound?" Your intuition may have hinted you that since all the Bell states can achieve $2\sqrt{2}$ violation under some settings, then surely a mixture of these Bell states can also saturate the Cirel'son' son bound. However, recall that the mapping eqn. (2.64) depends on more than just the choice of density operators and involves a careful selection of measurements as well. Hence for two qubits subspace the answer is no but surprisingly for higher dimensions the answer is yes [29]. Now lets examine why in the two qubits subspace, one cannot achieve maximum violation of $2\sqrt{2}$ with mixtures of Bell states eqn. (2.57)

Now consider the Bell states in terms of the correlation matrix T eqn. (2.67)

$$T(\Phi^+) := \mathrm{diag}(+,-,+) \tag{3.28}$$

$$T(\Phi^-) := \mathrm{diag}(-,+,+) \tag{3.29}$$

$$T(\Psi^+) := \mathrm{diag}(+,+,-) \tag{3.30}$$

$$T(\Psi^-) := \mathrm{diag}(-,-,-) \tag{3.31}$$

where they form a tetrahedron in the correlators space. Hence one can see that the T matrices gives the correlators function in terms of $\{\vec{a},\vec{a}',\vec{b},\vec{b}'\}$ and recall that the correlator for $|\Psi^-\rangle$ is given as $-\vec{a}\cdot\vec{b}$ which saturate the Cirel'son bound for settings

$$\vec{a}=\hat{z}, \quad \vec{a}'=\hat{x}, \quad \vec{b}=\frac{\hat{z}+\hat{x}}{\sqrt{2}}, \quad \vec{b}'=\frac{\hat{z}-\hat{x}}{\sqrt{2}} \tag{3.32}$$

and $|S|=|-2\sqrt{2}|=2\sqrt{2}$ is achieved. But for $|\Psi^+\rangle$ state, the settings that achieve $-2\sqrt{2}$ violation are just the relabelling of Bob's measurement settings:

$$\vec{a}=\hat{z}, \quad \vec{a}'=\hat{x}, \quad \vec{b}=\frac{\hat{z}-\hat{x}}{\sqrt{2}}, \quad \vec{b}'=\frac{\hat{z}+\hat{x}}{\sqrt{2}} \tag{3.33}$$

This implies that these two states do not share the same optimal settings that can achieve maximal violation of $|S|=2\sqrt{2}$. As a check, one can consider the necessary and sufficient condition eqn. (2.66) with $\rho=\mathrm{P}|\Psi^+\rangle\langle\Psi^+|+(1-\mathrm{P})|\Psi^-\rangle\langle\Psi^-|$,

$$T(\rho)=\mathrm{diag}(2\mathrm{P}-1,2\mathrm{P}-1,-1) \tag{3.34}$$

$$\max_{\mathcal{B}}|S(\rho)_{\mathrm{CHSH}}|=2\sqrt{1+(2\mathrm{P}-1)^2} \tag{3.35}$$

which gives a CHSH value of $2\rightarrow 2\sqrt{2}$ as P goes from $0.5\rightarrow 1$. Hence any mixtures of the Bell states in the two qubits subspace can never saturate the Cirel'son bound. But if the quantum state is in a mixture of orthogonal block-wise generalized singlets eqn. (3.3), one can use the block-wise measurement scheme eqn. (3.23) to obtain $2\sqrt{2}$ violation.

3.2 $\mathbb{C}^4 \otimes \mathbb{C}^4$ system numerical studies

As shown in above examples, if one restricts the black box source to just two qubits subspace, the ideal states that can achieve maximal violation are the Bell states, $\{|\Phi^{\pm}\rangle, |\Psi^{\pm}\rangle\}$. Also note that these Bell states do not form a convex set due to the fact that the Bell states are unique up to local unitaries. However, upon considering higher dimensions systems, there exists mixtures of generalized singlets that reside in orthogonal subspaces that can leads to a violation of $2\sqrt{2}$. This has indeed poise an impasse in estimating the source with reference to ideal states. Because it is the clear definition of idea states in the two qubits subspace that enables one to solve eqn. (3.6) and eqn. (3.9) by using spectral decomposition of Bell operator for qubits. Now it seems that the trace distance between the observed system and the closest ideal state may now be operationally blurred, i.e., It is not clear if there exists a mixture of generalized singlet that may give a better trace distance. It is also not known that for any given observed state, what are the quantum states that satisfy eqn. (3.6) or eqn. (3.9), as we seen in the restriction under two qubits studies, the closest ideal state is only one of the Bell states.

3.2.1 Conjectured family of ideal states

In general, let the states of both sub-systems Alice and Bob be represented by a set of local orthonormal bases $\{|a\rangle_i\}$ and $\{|b\rangle_i\}$ respectively. Hence the representation space of the composite system, any pure states can be represented by the Schmidt decomposition as

$$|\Psi\rangle = \sum_{i=0}^{3} c_i |a\rangle_i \otimes |b\rangle_i \tag{3.36}$$

It then possible to arrange the Schmidt coefficients such that they are real, positive and in descending order, $c_0 \geq c_1 \geq c_2 \geq c_3 \geq 0$. Following from the formulation of the problem, one can then try to define or determine the complete set of ideal states that satisfy the condition eqn. (3.27). Here the family of idea states for $\mathbb{C}^d \otimes \mathbb{C}^d$ is suggested to be of the form

$$\Phi_{ideal} := \sum_{k=0}^{d/2-1} \lambda_k |\Phi_k^+\rangle\langle\Phi_k^+| + \lambda_{d/2} |\Phi^+_{\{\theta_0,\theta_1,\cdots,\theta_{d/4-1}\}}\rangle\langle\Phi^+_{\{\theta_0,\theta_1,\cdots,\theta_{d/4-1}\}}| \tag{3.37}$$

where $\sum_{k=0}^{d/2} \lambda_k = 1$ and

$$|\Phi^+_{\{\theta_0,\theta_1,\cdots,\theta_{d/4-1}\}}\rangle := \sum_{m=0}^{d/4-1} \left[\cos(\theta_m)|\Phi_m^+\rangle + \sin(\theta_m)|\Phi_{m+1}^+\rangle\right] \tag{3.38}$$

with $|\Phi_m^+\rangle$ given as the generalized singlet

$$|\Phi_m^+\rangle = \frac{1}{\sqrt{2}} \left(|2m, 2m\rangle + |2m+1, 2m+1\rangle\right) \tag{3.39}$$

Fig. (3.2) is an illustration of a $\mathbb{C}^6 \otimes \mathbb{C}^6$ subspace where one can observe that the family of ideal states is just a convex combination of orthogonal subspace singlets of $\mathbb{C}^2 \otimes \mathbb{C}^2$ with an additional full rank maximally entangled state.

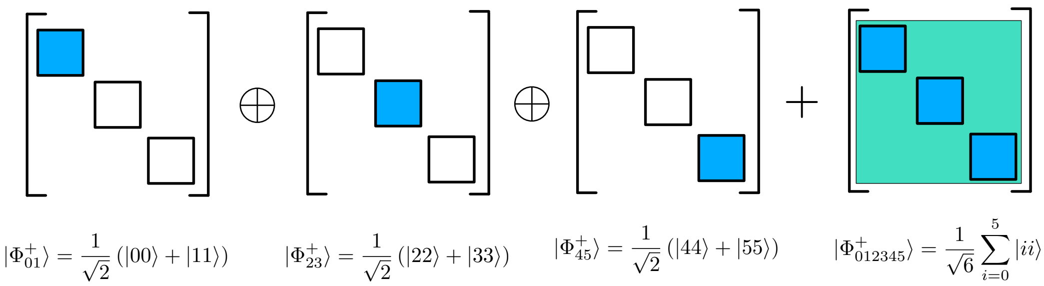

Figure 3.2: An illustration of how the family of ideal states looks like for a $\mathbb{C}^6 \otimes \mathbb{C}^6$ space. Note that the weights, λ_k are omitted for simplicity of illustration.

Now that we have defined the suggested family of ideal states for arbitrary bipartite states, one gets the following family under the restriction of $\mathbb{C}^4 \otimes \mathbb{C}^4$ subspace,

$$\Phi^{4\otimes 4}_{\text{ideal}} = \lambda_0 |\Phi_0^+\rangle\langle\Phi_0^+| + \lambda_1 |\Phi_1^+\rangle\langle\Phi_1^+| + \lambda_2 |\Phi_\theta^+\rangle\langle\Phi_\theta^+| \tag{3.40}$$

Next define the computation basis for both $\mathcal{H}_{\text{A}}$ and $\mathcal{H}_{\text{B}}$ as $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ s.t

$$|\Psi\rangle = a\,|00\rangle + b\,|11\rangle + c\,|22\rangle + d\,|33\rangle \tag{3.41}$$

where $a \geq b \geq c \geq d \geq 0$ and the pure ideal states are

$$\left|\Phi_0^+\right\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \; \left|\Phi_1^+\right\rangle := \frac{1}{\sqrt{2}}\left(|22\rangle + |33\rangle\right) \tag{3.42}$$

Also note that the following pure states also satisfy the condition eqn. (3.27)

$$\left|\Psi_0^+\right\rangle := \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \; \left|\Psi_1^+\right\rangle := \frac{1}{\sqrt{2}}\left(|23\rangle + |32\rangle\right) \tag{3.43}$$

however, if one defines the observed state to be in the form of eqn. (3.41) then the closest reference states are always of the form of $\{|\Phi_0^+\rangle, |\Phi_1^+\rangle\}$.

Using eqn. (2.40), one gets a quantum trace distance operator function that is dependent on the 8 parameters $\{a, b, c, d, \lambda_0, \lambda_1, \lambda_2, \theta\}$ and these 8 parameters are further reduced to 6 independent parameters upon invoking normalization conditions. Hence for a given quantum state $\{a, b, c\}$ the Mayers & Yao trace distance reads

$$\delta_{\mathrm{MY}}(|\Psi\rangle) = \min_{\lambda_0,\lambda_1,\theta} \frac{1}{2} \left\| |\Psi\rangle\langle\Psi| - \Phi_{\mathrm{idea}^{4\otimes 4}} \right\|_1 \tag{3.44}$$

3.2.2 Analytical extension of Horodecki's CHSH condition for certain bipartite mixed states

Next we need to compute the maximal possible violation achievable by $|\Psi\rangle$ and the only known solution is numerical optimization for maximal violation for any arbitrary bipartite states. If we consider only pure states, then eqn. (3.25) can be invoked as in the case of Bardyn et al [3]. Here, we provide an analytical solution for arbitrary even dimension bipartite state of the form:

$$\rho = \sum_k \mathrm{P}_k |\Psi_k\rangle\langle\Psi_k| \tag{3.45}$$

where $|\Psi_k\rangle$ are pure states of ordered form, eqn. (3.19). One can generalize the Gisin-Peres conjecture eqn. (3.23) and define the measurement scheme with the following observables:

$$A(\hat{a}) := a_x\Gamma_x + a_y\Gamma_y + a_z\Gamma_z \tag{3.46}$$

$$A(\hat{a}') := a'_x\Gamma_x + a'_y\Gamma_y + a'_z\Gamma_z \tag{3.47}$$

$$B(\hat{b}) := b_x\Gamma_x + b_y\Gamma_y + b_z\Gamma_z \tag{3.48}$$

$$B(\hat{b}') := b'_x\Gamma_x + b'_y\Gamma_y + b'_z\Gamma_z \tag{3.49}$$

the CHSH operator is then defined with the following unit vectors $\{\hat{a}, \hat{a}', \hat{b}, \hat{b}'\}$. Here we analytically optimize the CHSH operator with the given observables and ρ:

$$\max_{\{\hat{a},\hat{a}',\hat{b},\hat{b}'\}} \mathrm{tr}(\mathcal{B}\rho) = \max_{\{\hat{a},\hat{a}',\hat{b},\hat{b}'\}} (\mathrm{tr}[\rho(\hat{a}\cdot\vec{\Gamma}\otimes(\hat{b}+\hat{b}')\cdot\vec{\Gamma})] + \mathrm{tr}[\rho(\hat{a}\cdot\vec{\Gamma}\otimes(\hat{b}-\hat{b}')\cdot\vec{\Gamma})]) \tag{3.50}$$

let the unit vectors of Bob be related [26] by

$$\hat{b}+\hat{b}' := 2\mathrm{cos}(\theta)\hat{c} \tag{3.51}$$

$$\hat{b}-\hat{b}' := 2\mathrm{sin}(\theta)\hat{c}' \tag{3.52}$$

such that $\hat{c}$ and $\hat{c}'$ are orthogonal vectors.

$$\max_{\{\hat{a},\hat{a}',\hat{c},\hat{c}',\theta\}} \mathrm{tr}(\mathcal{B}\rho) = \max_{\{\hat{a},\hat{a}',\hat{c},\hat{c}',\theta\}} 2\left(\mathrm{tr}[\rho(\hat{a}\cdot\vec{\Gamma}\otimes\hat{c}\cdot\vec{\Gamma})]\mathrm{cos}(\theta) + \mathrm{tr}[\rho(\hat{a}'\cdot\vec{\Gamma}\otimes\hat{c}'\cdot\vec{\Gamma})]\mathrm{sin}(\theta)\right) \tag{3.53}$$

Here, one can optimize the Bell operator with respect to θ by using the bound eqn. (3.15)

$$\max_{\{\hat{a},\hat{a}',\hat{c},\hat{c}'\}} \mathrm{tr}(\mathcal{B}\rho) = \max_{\{\hat{a},\hat{a}',\hat{c},\hat{c}'\}} 2\sqrt{\mathrm{tr}[\rho(\hat{a}\cdot\vec{\Gamma}\otimes\hat{c}\cdot\vec{\Gamma})]^2 + \mathrm{tr}[\rho(\hat{a}'\cdot\vec{\Gamma}\otimes\hat{c}'\cdot\vec{\Gamma})]^2} \tag{3.54}$$

Similar in the formulation of maximal CHSH violation for two qubits by Horodecki et al [17], we let $k_{ij} := \mathrm{tr}(\rho\vec{\Gamma}_i\otimes\vec{\Gamma}_j)$ be the quantum correlation matrix, K. Hence we can maximize the operator by letting $\hat{c}$ and $\hat{c}'$ be the eigenvectors of K^TK and choosing $\hat{a}$ and $\hat{a}'$ to be unit vectors that is in the direction of $K\hat{c}$ and $K\hat{c}'$ respectively. Hence we obtain the maximal CHSH violation for arbitrary even dimension bipartite system by defining $L(\rho) := \max_{\{\hat{c},\hat{c}'\}} (|K\hat{c}|^2 + |K\hat{c}'|^2)$, hence

$$\max_{\{\hat{c},\hat{c}'\}} \mathrm{tr}(\mathcal{B}\rho) = \max_{\{\hat{c},\hat{c}'\}} 2\sqrt{L(\rho)}. \tag{3.55}$$

From the extension of the Horodecki [17] work to bipartite systems of eqn. (3.45), we could also recover the conjecture eqn. (3.25) given by Gisin & Peres [27]

$$K(|\Psi\rangle) = \mathrm{diag}(2ab + 2cd, -2ab - 2cd, 1) \tag{3.56}$$

$$L(|\Psi\rangle) = \mathrm{diag}((2ab + 2cd)^2, (2ab + 2cd)^2, 1) \tag{3.57}$$

$$\max\langle\Psi|\mathcal{B}|\Psi\rangle = 2\sqrt{1 + 4(ab + cd)^2} \tag{3.58}$$

The analytical result for states of eqn. (3.45) was compared with the general optimization of CHSH operator and was found to have difference of 10^{-4} on the average.

3.2.3 Numerical studies for F_{MY} and D_{MY} under the restriction of $\mathbb{C}^4 \otimes \mathbb{C}^4$

In addition to considering pure states, we also considered the noisy pure states given as

$$\rho_{\text{noisy}} := x|\Psi\rangle\langle\Psi| + (1 - x)\frac{\mathbb{I}}{16} \tag{3.59}$$

The Mayers & Yao trace distance, fideliy and maximal CHSH violation for 10000 random pure states and 30000 noisy states were computed simultaneously and data were loaded into a matrix that produced fig. (3.3) and fig. (3.4)

Fidelity distance measure for pure states and noisy states

One can see that the theoretical bound eqn. (3.26) is respected by the numerical results of both pure and noisy states and fig. (3.3) implies that noisy states under numerical verification cannot be a worse bound than the simulations(blue scatter plots) by the pure states. This is indeed interesting as it seems that with white noise, the fidelity can be improved. Another interesting point is that numerical fidelity for pure states is given by the two qubits result eqn. (3.18) (the black line) and the bound given by [3] eqn. (3.26)

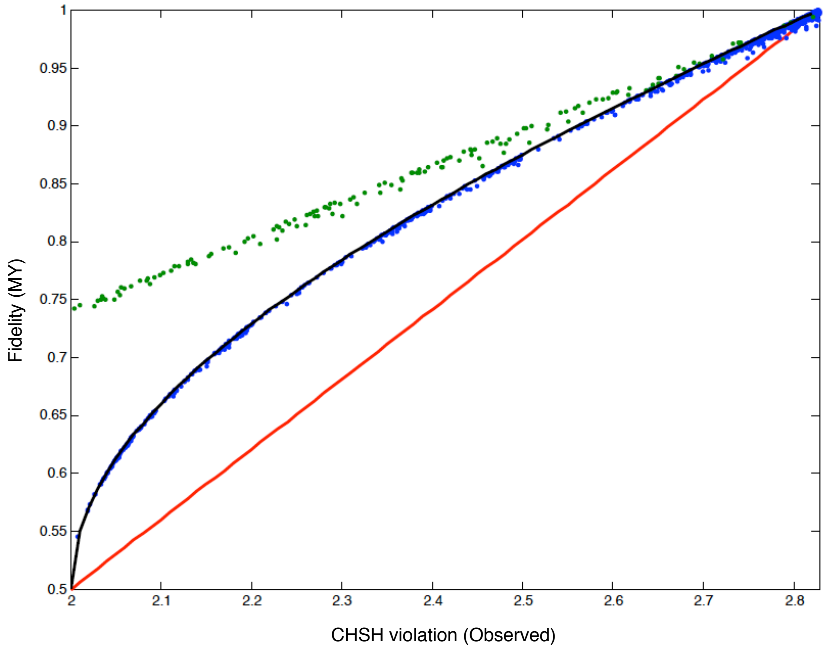

Figure 3.3: The (1) red coloured line indicates the theoretical bound eqn. (3.26) computed by Bardyn et al for pure states, qudits. (2) the black theoretical line indicate the Mayers & Yao fidelity for qubits eqn. (3.18). (3)The blue scatter plots refer to numerical simulation of random pure states of eqn. (3.41) and (3) green scatter plots indicate the numerical simulation of noisy states, eqn. (3.59)

appears to be not tight under this numerical studies. This result is indeed interesting as the numerical results suggest that the tighter bound is given by the qubits bound.

Trace distance measure for pure states and noisy states

Here, we compute the trace distance measure independently of the relation eqn. (2.49), but use the inequality between trace distance and fidelity eqn. (2.49) so that one can compare the theoretical bound with the numerical results: The analytical trace distance can be obtained after one does a direct substitution of eqn. (3.26) into eqn. (2.49):

$$\delta_{\text{MY}} \leq \sqrt{1 - \frac{1}{4(\sqrt{2}-1)}[S_{\text{obs}} + 2\sqrt{2} - 4]} \tag{3.60}$$

which gives the red colour bound in fig. (3.4) and for two qubits is given by the black line in fig. (3.4)

$$\delta_{\text{MY}} \leq \sqrt{\frac{1 - \sqrt{S_{\text{obs}}^2/4 - 1}}{2}} \tag{3.61}$$

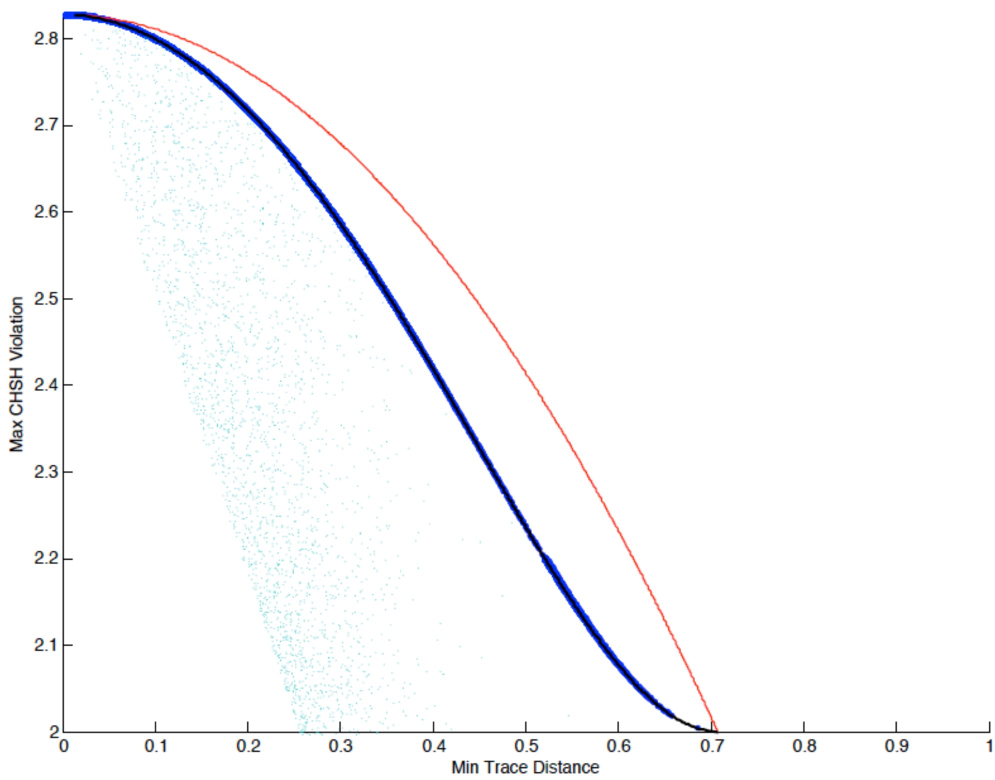

Figure 3.4: The (1) red coloured line indicates the theoretical bound eqn. (3.26) computed by Bardyn et al for pure states. (2) the black theoretical line indicate the Mayers & Yao fidelity for qubits eqn. (3.18). (3) The blue scatter plots indicate the numerical simulation of random pure states of eqn. (3.41) and (4) cyan scatter plots indicate the numerical simulation of noisy states, eqn. (3.59)

Here it appears that the computed bound (blue scatter plots) fig. (3.4) for trace distance of pure states is again given by eqn. (3.61). It was initially thought that the fidelity and trace distance measures may not have the same family of ideal states, but from this result it seems that they share the same family in $\mathbb{C}^4 \otimes \mathbb{C}^4$. It is also noted that the noisy states are bounded by the pure states, which is consistent with the result from the fidelity measure. The trace distance numerical results also agree with the results from fidelity measure that the two qubits bound may be a tighter bound than the one suggested by [3].

3.2.4 Numerical search for closest ideal states

The numerical studies for $\mathbb{C}^4 \otimes \mathbb{C}^4$ suggested that the Mayers & Yao fidelity by Bardyn et al for pure states is not a tight bound and now it appears that one only need to prove that for any arbitrary states, the bound eqn. (3.26) given by [Bardyn] is the lowest bound. Another interesting question is that given a density matrix with some observed violation of S_{obs}, what are the states that satisfy the δ_{MY}? i.e., ideal states that can give the minimum trace distance. It was found out that under the restriction of pure states, the closest ideal states can be mixed or pure and the converse is also true. In the following demonstration, instead of pure states we use a noisy maximally entangled state as an example.

$$\rho = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}\frac{\mathbb{I}}{16} \tag{3.62}$$

and again we use the family of ideal states as

$$\Phi_{\text{ideal}}^{4\otimes4} = \lambda_0|\Phi_0^+\rangle\langle\Phi_0^+| + \lambda_1|\Phi_1^+\rangle\langle\Phi_1^+| + \lambda_2|\Phi_\theta^+\rangle\langle\Phi_\theta^+| \tag{3.63}$$

now it seems that under the optimization (minimization) of δ_{MY}, there seems to be infinitely number of closest reference states! fig. (3.5) is a collection of all points that satisfy eqn. (3.27) and δ_{MY} but with different optimization points, i.e., different values of θ, λ_0 and λ_1. This result is amusing in the sense that if one considers only two qubits subspace, there is only one closest idea state but if one moves above two qubits dimensions, then one can get infinitely many ideal states. To get an idea on how this may affect the estimation, we consider the entanglement measure of the ideal states.

Negativity of ideal states

We introduce negativity measure by Vidal and Werner [33], which is a measure of entanglement for an arbitrary bipartite state ρ:

$$\mathcal{N}(\rho) := \frac{\|\rho^{T_A}\|_1 - 1}{2} \tag{3.64}$$

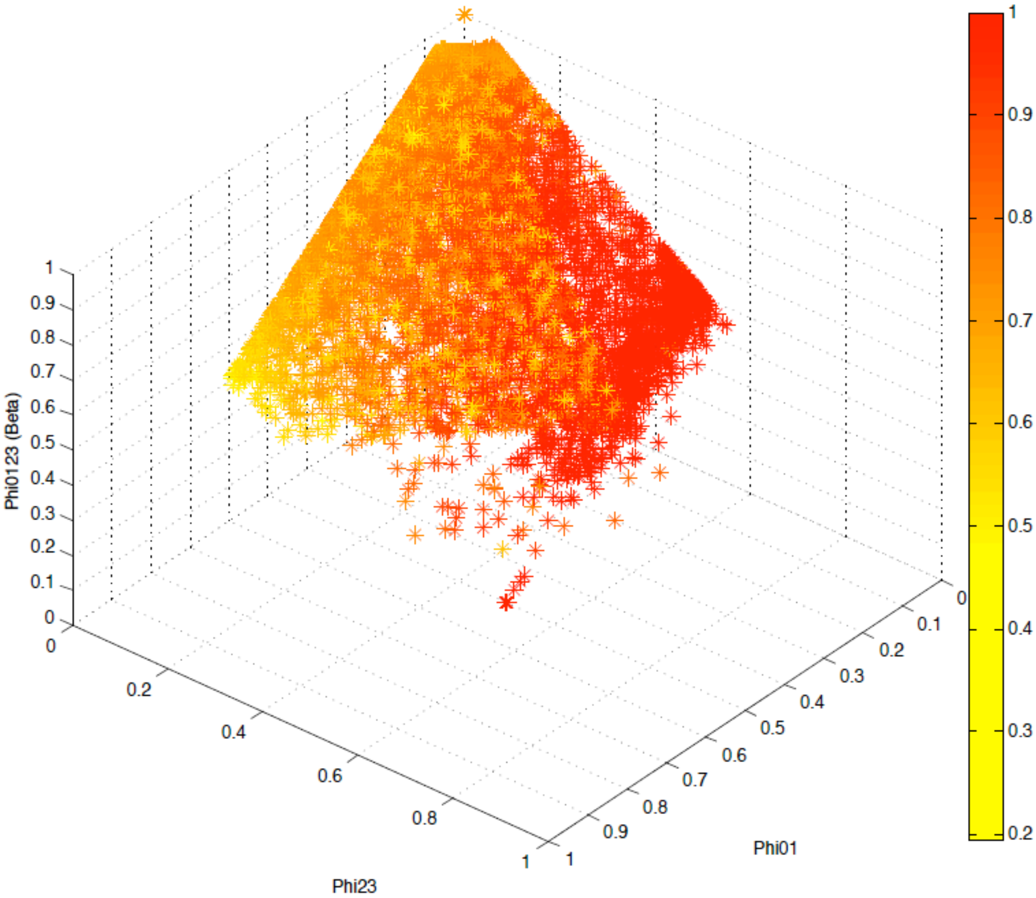

Figure 3.5: The axes are given by the classical distribution of $|\Phi_0^+\rangle$ for Phi01 axis, $|\Phi_1^+\rangle$ for Phi23 axis, $|\Phi_\theta^+\rangle$ for Phi1234 axis while the colour chart on the right side represent the value of $\cos(\theta)$. The data points plotted on the graph satisfy eqn. (3.27) and δ_{MY} and they all define the same trace distance.

which can be computed for the states corresponding to the axis of fig. (3.5).

$$\begin{aligned}\mathcal{N}(|\Phi_{0123}^+\rangle) &= |-0.25 \times 6| = 1.5\\ \mathcal{N}(|\Phi_{01}^+\rangle) &= |-0.5 \times 1| = 0.5\\ \mathcal{N}(|\Phi_{23}^+\rangle) &= |-0.5 \times 1| = 0.5\end{aligned} \tag{3.65}$$

where

$$
\begin{aligned}
|\Phi^+_{0123}\rangle &= \frac{1}{\sqrt{4}}\left(|00\rangle + |11\rangle + |22\rangle + |33\rangle\right) \\
|\Phi^+_{01}\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
|\Phi^+_{23}\rangle &= \frac{1}{\sqrt{2}}\left(|22\rangle + |33\rangle\right)
\end{aligned} \tag{3.66}
$$

now it is obvious that the entanglement measure, negativity of the ideal states that are closest to the observed state via the statistics can range from 0.5 to 1.5. It is must be emphasised that such an entanglement measure scale with the rank of the density operator, i.e., if the higher the dimension of a maximally entangled quantum state, the higher the negativity is as the negativity is the summation of negative eigenvalues, $|\sum_i \mu_i|$. The demonstration of the above example hinted the implication of having many closest ideal states instead of just an unique closest state. What this entails is that as much as one tries to estimate the quality of the entanglement of the black box source, the minimum amount of knowledge (specifications) one has prior to the estimation is sufficient to detect entanglement but this may result in a weak bound as the dimension scales higher. Can more be known via just the statistics? Yes, one can indeed put a lower bound on the dimension of the unknown system [34].

3.2.5 Comments on numerical studies

It appears that much numerical studies are needed to gain better idea of how the ideal states can be characterized. In the current work of the project, it was found that family of ideal states suggested in eqn. (3.37) is the optimal one in the studies of $\mathbb{C}^4 \otimes \mathbb{C}^4$ subspace. It was also this family that suggested that there are infinitely many closest ideal states which again demonstrated the difficulty of working in higher dimensions. But nevertheless, one can always adapt the proof by Bardyn et al and try to prove that the lower bound given by the them is the lowest bound under arbitrary bipartite states. It also appears that the bound computed for $\mathbb{C}^4 \otimes \mathbb{C}^4$ gives the same bound as the one for two qubits in fig. (3.3) and fig. (3.4). One could further try to find the numerical bounds for $\mathbb{C}^{>4} \otimes \mathbb{C}^{>4}$ and try to check if indeed the bound given by the two qubits case is the tightest bound for arbitrary bipartite states.

Chapter 4

Teleportation channel

4.1 Introduction

In §(2.4.4), we introduce the quantum teleportation protocol that allows one to send an unknown state (at Alice laboratory) to another location (Bob laboratory) in a single run of the protocol. This single run requires a distributed entangled pair and 2 classical bits of communication. It is also emphasized that the communication required during the protocol is just 2 classical bits and no qubits are communicated. It seems that the presence of an entangled pair that is distributed between Alice and Bob laboratories complements the rest of the information in a successful teleportation. So one in general can view the teleportation protocol as a channel that receives an unknown state and exactly output the unknown state between two locations. It is also worth noting that the distance between the laboratories has no bearings on the communication requirements. In this chapter, I introduce the idea which all the quantum apparatus involved in a standard quantum teleportation protocol now are black boxes. §4.2 introduces the notion of average fidelity which will be a figure of merit for the teleportation channel performance, in §4.3 the quantum teleportation black box statement and protocols will be defined and lastly §4.4 summarizes the current research progess in this direction.

4.2 Figure of merit for quantum teleportation channel

Before the teleportation is conducted, the mathematical description of the state of the composite system is given by

$$|\Psi\rangle_{ABC} = |\phi\rangle_A \otimes |\Phi_{\rm res}\rangle_{BC} \tag{4.1}$$

where system A, B and C refer to unknown state, Alice side of entangled pair and Bob side of entangled pair respectively. After a single run of the teleportation protocol, Bob will have the state $|\phi\rangle$ exactly in his laboratory and the entangled pair $|\Phi_{\rm res}\rangle_{BC}$ will no longer be distributed between the both parties, i.e., they lose their quantum teleportation resource. In general, to distribute a perfect maximally entangled state is not that simple as one has to face with transmission losses, imperfect generators, human mistakes, etc. Suppose that one has finally m copies of maximally entangled states and wishes to conduct teleportation m times, what is the accuracy of these runs of teleportation? This is indeed a very valid question as one has to have some form of estimation to quantify how accurate this teleportation channel is.

Average teleportation fidelity

A natural measure of a teleportation channel is the fidelity of the output state with respect to the input state. This definition is given as [35–37]

$$\bar{F}_{\rm T} = \int d\phi \rho(\phi) \sum_k {\rm P}_k \langle\phi|\rho_{{\rm Bob},k}|\phi\rangle \tag{4.2}$$

where P_k is the probability of getting k outcome from the BSM measurement and $\rho_{{\rm Bob},k}$ is the final density matrice after Bob does his unitary operation conditioned on the outcome k. The P_k is given by

$$\mathrm{P}_k = {\rm tr}\left[(P_k \otimes \mathbb{I})(|\phi\rangle\langle\phi|_A \otimes |\Phi_{\rm res}\rangle\langle\Phi_{\rm res}|_{BC}\right] \tag{4.3}$$

It is then obvious that the average fidelity is 1 for perfect quantum teleportation, i.e., perfect Bell state measurements, entangled source, unitary operations and pre-established protocol based on the resource (the knowledge of the resource is essential). Interestingly, Gisin [36] 1995 showed an example of a classical teleportation protocol (not shown to be the optimal scheme) that can achieve an average teleportation fidelity of ~ 0.87. In 1999, Cerf, Massar and Gisin [38] showed that with a more elaborated classical scheme, one can achieve an average teleportation fidelity of ~ 0.95 and Toner & Bacon [39] in 2003 showed that one can simulate quantum correlation of a modified teleportation scenario with a local hidden variables model and communication of 2 classical bits. However it is important to emphasize that if one is restricted to only quantum teleportation systems, [37,40] showed that with just local quantum states, the average fidelity is upper bounded by $\frac{2}{3}$ and $\frac{2}{d+1}$ for qubits and pure qudits states respectively. In fig. (4.1), only results from

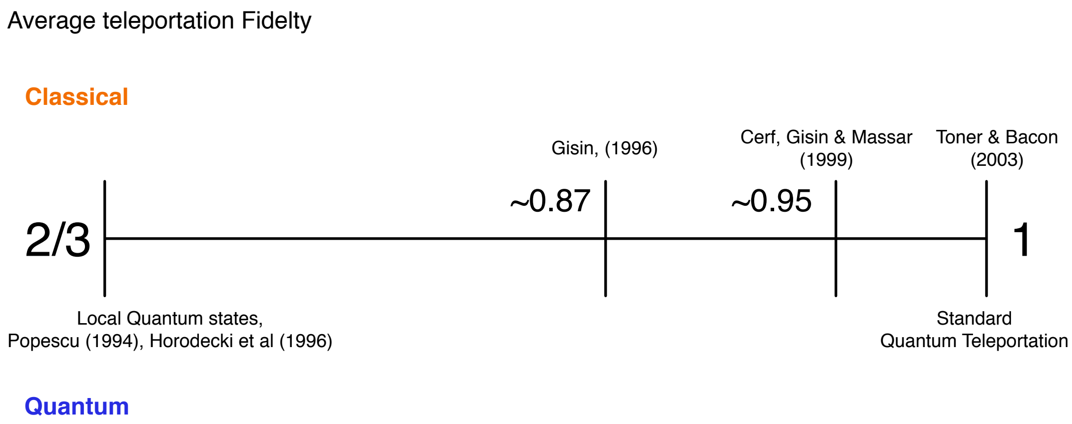

Figure 4.1: A pedagogical illustration of the average teleportation fidelities that can be achieved by black box classical teleportation.

2/3 is considered as the main ideal is to detect the entanglement involved in quantum teleportation channel. However, note that in this scale all the results of the classical teleportation schemes are assumed to be adapted to black box scenarios. That is the users of the black box teleportation cannot know anything more than just the basic instructions to perform the teleportation. That is to say, Alice will key in a description of the "teleported" state into the black box in her laboratory instead of feeding in an unknown quantum state. However as one can notice in fig. (4.1) that the Toner & Bacon model is indistinguishable from the standard quantum teleportation results. So it seems that if one just uses the average fidelity alone as the figure of merit to estimate the black box teleportation scenario, it is impossible to distinguish between the classical and quantum schemes. How one has to bear in mind that in quantum teleportation, entangled pairs are involved and these quantum states have quantum correlations that cannot be simulated by classical resources, i.e., Bell's inequalities are non-local correlations witnesses. In the following section, black box teleportation will be introduced and a simple model of classical teleportation that can achieve fidelity of 0.75 will be demonstrated.

4.2.1 Considering quantum integrity of the teleportation channel

In most of the above pioneering works, Bob always receives the 2 classical bits and conducts the the necessary operations to achieve the stated average fidelity. However in a black box scenario, this part of the protocol will be discarded, that is Bob is to measure the supposedly teleported state in some basis, $\vec{b}$. Toner & Bacon showed that even if Bob instead of performing unitary operations does a projective measurement on his side of "quantum" resource, it is possible to simulate the quantum correlation with two classical bits and a local hidden variable model. Now if one denies the black boxes to communicate, then one may be able to distinguish between quantum and classical teleportation systems. So in principle, assume the black boxes are non-signaling and conduct Alice and Bob's laboratories at space-like separated locations. This will in general prevent all communications between the two black boxes. For each run of the teleportation, Alice and Bob will perform their measurement before the black boxes can communicate and meet up at the end of the experiment to compute certain conditional probabilities that will reveal certain classical models. But before we move on to the details on how to reveal classical models, it is important to mention that the knowledge of the specification of the black box source is assumed and the entire scenario is based on two level systems, qubits.

Distribution and specifications of black boxes

Let Alice and Bob be distributed with three black boxes as described in fig. (4.3):

1. Alice's laboratory: Allows the input description of a qubit by $\vec{m}_A$ (supposedly to

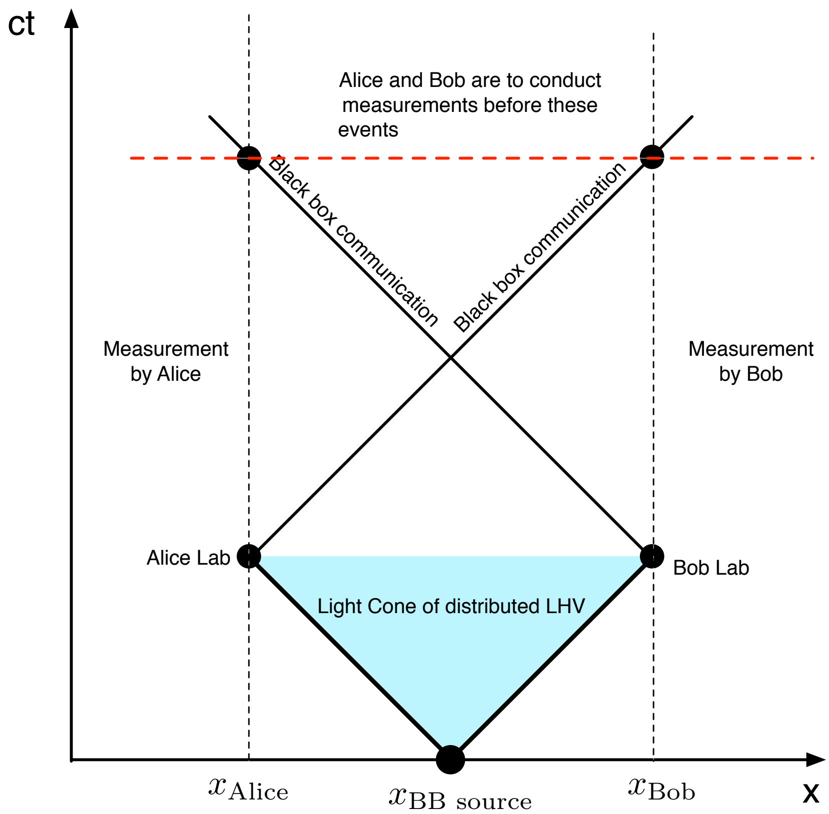

Figure 4.2: A space-time diagram of how to prevent the black boxes to communicate with each other, so as to rule out the black boxes attempt to simulate quantum correlations with aid of classical communication.

be teleported) and Bell state measurement that gives four outcomes $k \in \{0, 1, 2, 3\}$

2. Bob's laboratory: A black box that accepts a measurement vector $\vec{b}$ and upon measurement gives two outcomes, $\{+, -\}$.

3. Source: A black box that supposedly admits pairs of entangled pairs.

4. Instruction manual: to perform quantum teleportation, one requires a reference table that translate the two bits received from Alice via the classical channel to the appropriate unitary operations to recover the teleportated state.

In addition to the specification, assume that the rotation matrices are given to Alice and Bob. The black box vendor could in general not provide Alice and Bob with the

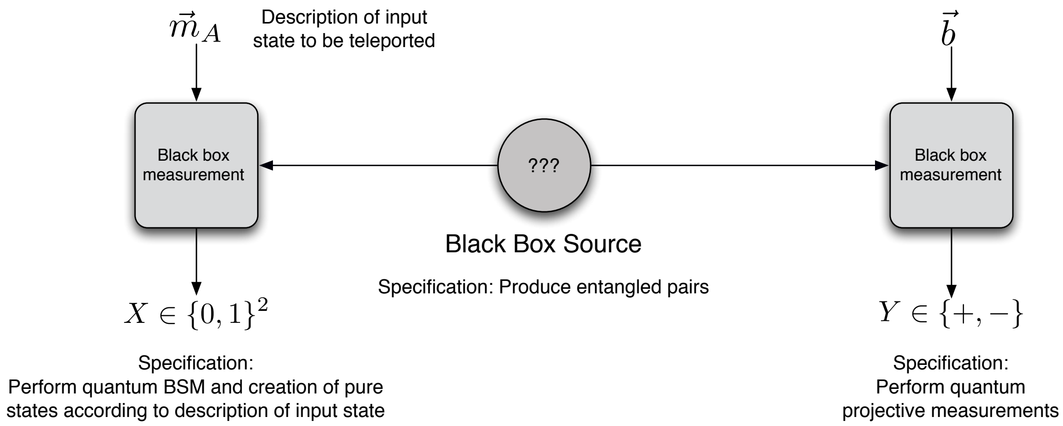

Figure 4.3: An illustration of a black box teleportation setup.

information of the rotation matrices, and the lack of this information may lead to the user being unable to distinguish between the black box teleportation statistics and quantum teleportation statistics. However, all hope is not lost as one could check for the non-locality of the channel and in general apply bounds [37] for quantum systems if non-locality is detected. Nevertheless, we now present an example of how can one with the knowledge of the rotation can estimate the presence of a classical model that gives average fidelity of 0.75.

4.3 A simple classical protocol

Suppose a (young and inexperienced) vendor design a black box source such that it only produces a classical vector, $\vec{V}^{\text{cl}} := \hat{z}$ and wish to use this black box source to simulate quantum teleportation with two other black boxes as in fig. (4.3). He then revealed his classical protocol to his investors and try to convince them that he can achieve average teleportation fidelity that is better than local quantum states and he doesn't require any entangled states as a resource. Of course, he makes all of the potential investors sign the non-disclosure forms for commercial reasons. The young vendor defines his protocol in the following:

1. The black box will always produce $\vec{V}^{\text{cl}} := \hat{z}$ and distribute them to Alice and Bob's black boxes.

2. Alice's black box will then receive Alice's description of her input state, $\vec{m}_A$ and

computes the scalar product between the $\vec{V}^{\mathrm{cl}}$ and $\vec{m}_A$. This function is defined as the local response function of the black box. The function will output only the sign of the result and is given as $q = \mathrm{sgn}(\vec{V}^{\mathrm{cl}} \cdot \vec{m}_A)$. If $q \geq 0$ then Alice black box will output $00 \Rightarrow \mathbb{I}$ or $01 \Rightarrow \sigma_z$, else $q < 0$ output $10 \Rightarrow \sigma_x$ or $11 \Rightarrow \sigma_y$. See fig. (4.4)

3. Now Bob receives the 2 classical bits from Alice, and perform the flip operation that takes $\vec{V}^{\mathrm{cl}}$ to the hemisphere that $\vec{m}_A$ is residing in.

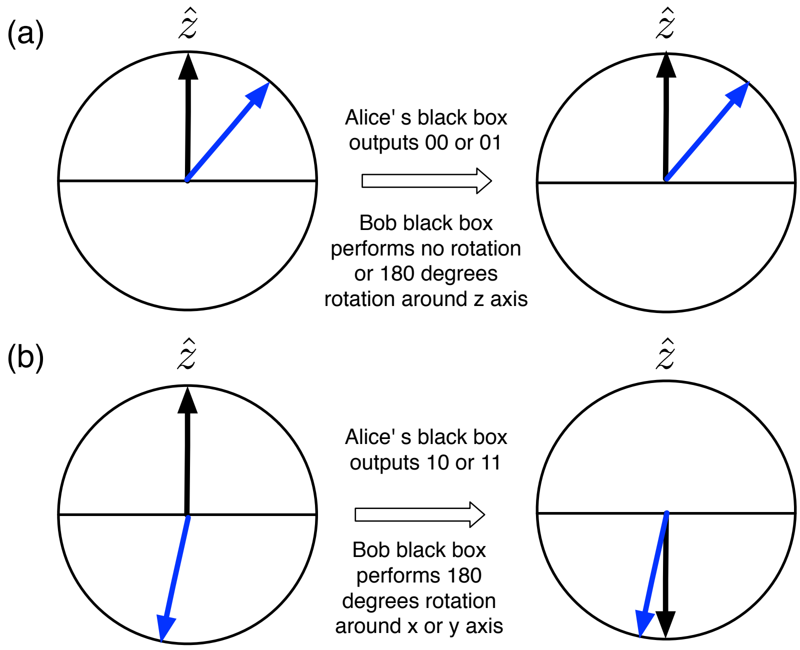

Figure 4.4: A 2D illustration of the simple classical protocol. (a) shows a situation where Alice keys in a $\vec{m}_A$ such that it is in the positive hemisphere of $\vec{V}^{\mathrm{cl}}$. Then the black box at Bob's lab does nothing or just 180 degrees rotation about $\hat{z}$ axis. (b) shows that Alice keys in a $\vec{m}_A$ in the negative hemisphere and the 2 bits communicated to Bob who will perform a 180 degrees rotation around $\hat{x}$ or $\hat{y}$ axis.

So at the end of each protocol, Bob will have a vector $\vec{m}_b = q\hat{z}$ which forms a hemisphere that contains $\vec{m}_A$. The average teleportation fidelity can be computed as

$$\bar{F}_{\mathrm{T}} = \frac{1}{2}\left[1 + \int d\rho(\vec{m}_A)\vec{m}_A \cdot \vec{m}_B\right] \tag{4.4}$$

with $d\rho(\vec{m}_A)$ the uniform measure. In spherical coordinates, $\vec{m}_A = \sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z}$. Therefore the average fidelity reads

$$\begin{aligned}\bar{F}_{\mathrm{T}} &= \frac{1}{2}\Big[1+\frac{1}{4\pi}\int_0^{\pi/2} d\theta\sin\theta\int_0^{2\pi} d\varphi\,\cos\theta \\ &\quad +\frac{1}{4\pi}\int_{\pi/2}^{\pi} d\theta\sin\theta\int_0^{2\pi} d\varphi\,(-\cos\theta)\Big]\end{aligned} \tag{4.5}$$

which gives

$$\bar{F}_{\mathrm{T}} = 0.75 \tag{4.6}$$

Now the young vendor highlighted that in quantum teleportation if one uses local quantum states, the $\bar{F}_{\mathrm{T}} \leq 2/3$ and hence his idea is amazing and buyers of the teleportation system will not suspect that his system is totally classical. Unfortunately, one of his investors is a principal investigator(in entanglement theory) of Centre of Quantum Technologies in Singapore and he doesn't believe in the young vendor's claims that it is foolproof, in addition this principal investigator claims that he can even estimate the vendor choice of $\vec{V}^{\mathrm{cl}}$ if the knowledge of rotation is revealed to him. Here is the principal investigator solution:

1. Now after the teleportation, Alice and Bob meet up and compare the statistics they obtained from the experiment. Bob can then filter his $\pm$ outcomes according to the rotations and the choice of Alice's $\vec{m}_A$.

2. Bob will then notice that for every $\vec{m}_A$ that is specified in the upper hemisphere, the rotation are always given as identity rotation or π rotation about $\hat{z}$ while in the lower hemisphere, the rotations given by Alice are always π rotation about $\hat{x}$ or $\hat{y}$. Hence after a number of runs of the teleportation, one can indeed observed that the black box has a fixed classical vector in the $\hat{z}$ direction.

The young vendor appears shell-shocked and decided to come back with another proposal to convince the investors and the smart principal investigator that he has a way of doing classical teleportation without entanglement as a resource.

4.4 An elaborated classical protocol

After much discussions with experts (people who don't like quantum information science) and refinement of his protocol, he comes before the investors once again and claims that he found a much better protocol [36] that can achieve average fidelity of ~ 0.87.
Here, the black box source instead of emitting maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ now distributes a set of vectors with some distribution (for simplicity, the distribution of the vectors is uniform).

$$\vec{V}_k \quad k \in \{0,1,2,3\} \tag{4.7}$$

where the set of vectors is defined as

$$\vec{V}_0 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \tag{4.8}$$

$$\vec{V}_1 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \tag{4.9}$$

$$\vec{V}_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \tag{4.10}$$

$$\vec{V}_3 = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \tag{4.11}$$

One can then check that the scalar product between any two vectors $\vec{V}_i \cdot \vec{V}_j = -1/3$ for $\forall\ i \neq j$. So the source for any single run will distribute any pairs of the vectors to both Alice and Bob with uniform probabilities, i.e., 1/4 of the time will be $\{\vec{V}_0^A, \vec{V}_0^B\}$.
Inside Alice's laboratory, Alice can enter her input state $\vec{m}_A = \cos\phi_A \sin\theta_A \hat{x} + \sin\phi_A \sin\theta_A \hat{y} + \cos\theta_A \hat{z}$. The Bell state measurement black box will then receive the $\vec{m}_A$ and as such an information is classical, the machine may then store the data in a memory and then perform the following computation:

$$\vec{V}_{\max} = \max_k (\vec{m}_A \cdot \vec{V}_k) \tag{4.12}$$

Where the maximum of the inner product is given by the optimal classical vector. The BSM black box will then do a comparison check to see which rotation operation will bring the given $\vec{V}_j$ to the $\vec{V}_{\max}$. The rotation operations are then given by:

$$R_0(\pi) = \mathrm{diag}(+,+,+) \tag{4.13}$$

$$R_1(\pi) = \mathrm{diag}(+,-,-) \tag{4.14}$$

$$R_2(\pi) = \mathrm{diag}(-,+,-) \tag{4.15}$$

$$R_3(\pi) = \mathrm{diag}(-,-,+) \tag{4.16}$$

For example, if the given $\vec{V}_j$ is the same as the $\vec{V}_{\max}$ then the suitable operation will be R_0. Hence Alice will recording the set of $\vec{m}_A$ and r for every single run of the teleportation. This set of information will then be used to compute the average fidelity of the teleportation channel at the end of the experiment.

It is then shown by [36] that the average fidelity of such a protocol is given by:

$$\bar{F}_{\mathrm{T}} = \frac{1}{2} + \sqrt{\frac{3}{2}}\frac{\arctan\sqrt{2}}{\pi} \simeq 0.87 \tag{4.17}$$

Now it is the principal investigator turn to be surprised that one can achieve an average teleportation fidelity of such value. The young vendor now appears very happy that the investors seems absolutely convinced that his classical protocol is an excellent business venture.

Principal investigator has the last laugh

Among the midst of laughters and congratulatory comments, the principle investigator raises up his hand and claims that one can still find a loop-hole in such a protocol and may estimate if the teleportation system uses a fixed distribution of classical vectors. Here is how it goes: Instead of Bob doing rotation operations, Bob will now do a measurement on his side of the supposedly entangled pair as shown in fig. (4.2). Bob will input a measurement vector $\vec{b}$ that will produce two outcomes $\{+,-\}$ with some probability distribution conditioned on $\vec{b}$ and $\vec{V}_j$ for every run of the system. The distinct difference from the standard quantum teleportation scheme here is that Bob instead of a unitary operation conditioned on the 2 classical bits from Alice, he performs measurement without any information from Alice now. This in principle will leave the state in the eigenstate of the measurement device under the quantum setting, given by

$$\rho_{\mathrm{Bob}} = \frac{1}{2}\left(\mathbb{I}_{2\otimes 2} \pm \vec{b}\cdot\vec{\sigma}\right) \tag{4.18}$$

while in the classical protocol, he will get the vector

$$\vec{m}_b' = \vec{V}_j \tag{4.19}$$

and the black box at Bob's laboratory can be defined by a local response function that takes in $\vec{b}, \vec{V}_j$ only and produce some distributions $\mathrm{P}(\pm|\vec{b}, \vec{V}_j)$.

After Alice's list of 2 classical bits and vectors $\vec{m}_A$

Now usually Alice sends Bob the 2 classical bits of information that corresponds to the necessary rotation R to recover the orignal state. However, Bob does not apply the rotation as in the standard quantum teleportation scheme but makes use of these information with his list of $\vec{b}$ vectors to compute the probability distributions pertaining to the supposedly quantum settings.

$$\mathrm{P}(\pm|\vec{b}, \vec{m}_A, R) = \frac{1}{2}\left(1 \pm \vec{b}\cdot R\vec{m}_A\right) \tag{4.20}$$

Next, Bob will compute the trace distance between these two set of probability distributions to ascertain the distinguishibility of the ideal and current teleportation channel.

$$\begin{aligned} D(\Lambda_Q, \Lambda_C) &= \frac{1}{2}\int d\rho(\vec{m}_A)\int d\rho'(\vec{b})\,[|\mathrm{P}(+|\vec{b}, \vec{m}_A, R) - \mathrm{P}(+|\vec{b}, \vec{V}_j) \\ &+ |\mathrm{P}(-|\vec{b}, \vec{m}_A, R) - \mathrm{P}(-|\vec{b}, \vec{V}_j)|] \end{aligned} \tag{4.21}$$

$$D(\Lambda_Q, \Lambda_C) = \frac{1}{2}\int d\rho(\vec{m}_A) \int d\rho'(\vec{b})\, |\vec{b} \cdot R\vec{m}_A - g(\vec{b}, \vec{V}_j)| \tag{4.22}$$

Where $g(\vec{b}, \vec{V}_j)$ is the local response function of Bob's black box measurement that seeks to give the best (minimum) trace distance between the ideal teleportation channel and the classically simulated teleportation channel. In general, these family of local response functions can be of any form but only need to maps to the real interval $[-1, 1]$: $-1 \leq g(\vec{b}, \vec{V}_j) \leq 1$. As an example, a simple choice will be the scalar product function between the vectors

$$g(\vec{b}, \vec{V}_j) = \vec{b} \cdot \vec{V}_j \tag{4.23}$$

If the young vendor uses the above local response function eqn. (4.23) then the trace distance between the two channels is ~ 0.375. However Alice and Bob can go one more step further and try to estimate the classical distributions. That is, Bob filters out the rotation R and $\vec{m}_A$ for a given $\vec{b}$, he will observe the following distribution fig. (4.5), This method of distinguishing classical black box teleportation from the actual quantum teleportation is possible if the rotation matrix is given to the user to verify their statistics. Now it seems that if the vendor refuse to reveal the rotation matrix information, one can in general not buy the system since one cannot have the minimum information to verify the quantum integrity of the teleportation channel. However, one could in general derive an teleportation inequality [41] for the teleportation channel and determine if the system involves non-locality.

4.4.1 Comments on Black box teleportation

Under the full black box scenario whereby the users are not given the information of the rotation matrix, it is totally impossible to distinguish between the statistics of the black box teleportation and quantum teleportation. However if the rotation matrix is provided, one could reveal the classical model behind the source as shown in the two examples. Another approach that can reveal information about the black box source is the use of Bell's inequalities. Zukowski [41] derived an inequality that measures the non-locality part of the teleportation channel and this in principle is different from the

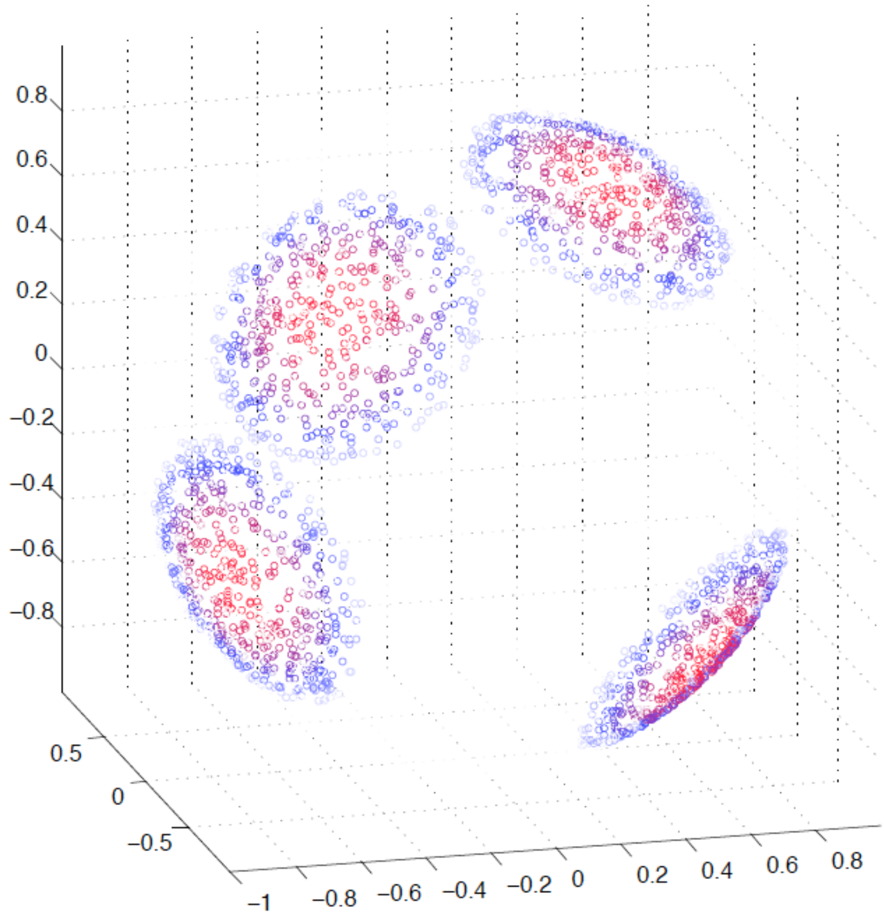

Figure 4.5: A 3D scatter plot of the statistics: The red plots are defined for $P_{Y|\vec{b},\vec{V}}(+) > 0.95$ and blue plots are defined for $0.95 \geq P_{Y|\vec{b},\vec{V}}(+) \geq 0.9$.

standard approach of Bell's inequalities. However, this model is yet to be studied in detail and work is ongoing to reformulate it to CHSH equivalence.

Chapter 5

Discussion

5.1 Black box state estimation with CHSH

In chapter 3, we use quantum correlation, an unique property of quantum systems to put a bound of how far the observed black box (or explicity the experimental statistics) is from the ideal states. In true nature, one is not trying to do a full quantum tomography here but rather is to give an estimation on the distinguishability between the statistics of an unknown black box setup and a genuine quantum entangled source. In doing so, one has to have the knowledge of the families of ideal states eqn. (3.27) so that meaningful estimation can be stated. To see why consider the following situation:
Suppose the observed CHSH violation is ~ 2.2, then one can compute lower bound the Mayers & Yao fidelity with this number and obtains ~ 0.62. But in general, this bound is valid but is no longer tight if the closet ideal state is a mixture of Bell states. In the analytical bound given by eqn. (3.26) under the restriction of pure states, assumes that the closest ideal state is always pure and this is yet to be proved to be true. However, in our numerical studies of $\mathbb{C}^4 \otimes \mathbb{C}^4$ we found that the closest ideal state can be both pure or mixed. For mixed states, we showed an example in fig. (3.5). So it seems that it is still reasonably safe to assume that the closest ideal states under the restriction of pure states are pure, at least in $\mathbb{C}^4 \otimes \mathbb{C}^4$. But one should prove this or even find an counter example that for a given pure state, the closest ideal state is exclusively a mixed state. In addition, we derive an analytical result for the optimization of CHSH inequality for

arbitrary bipartite states of even dimensions. This result is able to recover the Gisin & Peres conjecture eqn. 3.23 and eqn. 3.25 and it seems to lifted the requirement that the pure states are to be given in form of eqn. (3.19). However, much numerical studies must be done to justify these claims.

The numerical results also suggested that the bound given by eqn. (3.26) is not tight and the suggested analytical bound is the bound achieved by two qubits. This could be due to the fact that in the optimization we were considering a much larger family of ideal states than the authors in [3]. However bear in mind that the pure states are equally as close as to the mixed states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ and one in general should be able to improve on the bound given by [3] and conduct higher dimensions studies.

5.2 Black box state estimation with teleportation performance

In the problem of black box state estimation, we tried to shift the figure of merit to average teleportation fidelity and to gain some insights into black box teleportation scenario. It is well known that to transmit two real numbers, one needs an entangled pair and two classical bits of communication. This protocol is achieved by the quantum teleportation channel [23]. So how faithfully can one communicates the two real numbers to another space-like separated laboratory? This measure is given by the average teleportation fidelity, eqn. (4.2). However as much as we wanted to use the average teleportation fidelity to estimate the channel dependency on the entangled quantum resource, it seems that this figure of merit is saturated by the Toner & Bacon model [39] under the black box context. In their paper, they gave an example of a classical hidden variable model aided with two bits of classical communication can simulate the quantum correlations or more precisely the average teleportation fidelity. Bear in mind that under this assumption of black boxes, the information of rotations is not revealed to the users. Thus this is indeed a full black box situation. Nevertheless, we proceeded with the assumptions that the vendor will provide the users with the knowledge of the rotation matrix and indeed we showed two examples whereby one can use their statistics to estimate the classical hidden variable model.

5.3 Results and original works

Here are the list of original works and results from the eight months long research work.

5.3.1 State estimation

1. Conduct numerical studies of Mayers & Yao distances measures: fig. (3.4) and fig. (3.5) (1) suggested that the theoretical bound given by Bardyn et al [3] is not tight but is still the lowest bound for fidelity and uppermost bound for trace distance . (2) The numerical simulation of pure states are always given by the two qubits theoretical bound even under independent distance optimization (fidelity and trace distances are computed independently). (3)

2. Extension of Horodecki et al [17] conditions for two qubits to certain even dimensions bipartite states, eqn. (3.45). Comparison with general optimization of CHSH operators showed that the difference between the numerical optimization and analytical solution is $\sim 10^{-4}$.

3. Numerical search for closest ideal states revealed that the closest ideal states can be infinitely many. This is in contrast with the two qubits result. An example was shown in fig. (3.5)

5.3.2 Black box teleportation

1. Proposed a protocol to discriminate between black box teleportation and true quantum teleportation. This protocol requires the full knowledge of the rotation matrix. With respect to Gisin's model [36], we applied the protocol and numerically obtain an average of ~ 0.375 for the trace distance between the Gisin's model and the genuine teleportation channel. In addition, our protocol allows the user to estimate the distributions of the classical local hidden variables as seen in fig. (4.5).

5.4 Future work

In the context of black box physics as we have seen in chapter 3, one can estimate the quality of the entangled source with just the observed CHSH violation. In doing so the errors of the measurement apparatus are basically channeled to the black box source. This is an assumption imposed by the formulation of the problem. That is to say we assume the measurement devices are optimal and error free, so that we can formulate black box in the worse case scenario. This method of state estimation does not ends with just the targeted black box source but it may be extended to other quantum apparatus as well.

Example of quantum apparatus estimation

For example, consider the entanglement swapping scenario fig. (5.1): There are two independent black box sources which has been characterized with the Bardyn et al approach. That is to say, the black box measurements are optimal/perfect measurement schemes for the black box sources. As a start, consider the tensor product of the two pure non-maximally entangled source:

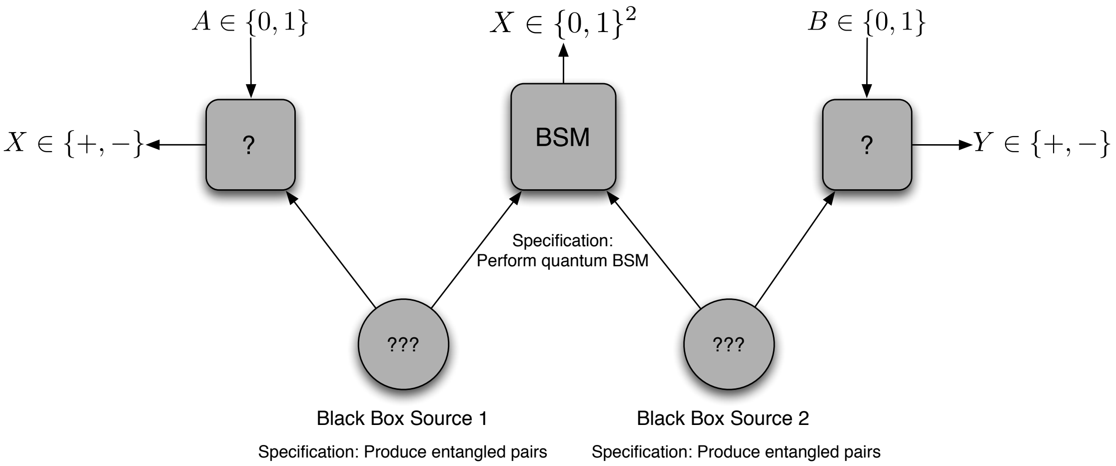

Figure 5.1: An illustration of black box swapping: In this scenario, one has already characterized the two independent black box sources with the same black box measurement devices.

$$|\Psi\rangle_{ABCD} = [\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle]_{AB} \otimes [\cos(\beta)|00\rangle + \sin(\beta)|11\rangle]_{CD} \qquad (5.1)$$

and now Bell state measurements (perfect) are made on the side B, C and the qubits A, D are projected onto the corresponding states:

$$|\Phi\rangle_{AD}^{\pm} = \frac{\cos(\alpha)\cos(\beta)|00\rangle \pm \sin(\alpha)\sin(\beta)|11\rangle}{\sqrt{\cos(\alpha)^2\cos(\beta)^2 + \sin(\alpha)^2\sin(\beta)^2}} \tag{5.2}$$

$$|\Psi\rangle_{AD}^{\pm} = \frac{\cos(\alpha)\sin(\beta)|01\rangle \pm \sin(\alpha)\cos(\beta)|10\rangle}{\sqrt{\cos(\alpha)^2\sin(\beta)^2 + \sin(\alpha)^2\cos(\beta)^2}} \tag{5.3}$$

Note that the entangled pair between A and D can never have a higher CHSH violation more than the lesser of the two independent sources. That is to say, the CHSH violation for the particles that never interacted is bounded by the $S_{\text{obs}}^{AD} \leq \min(S_{\text{obs}}^{AB}, S_{\text{obs}}^{CD})$. So in principle, if the measurement apparatus and bell state measurement are perfect, then the observed CHSH violation of the final pair satisfy the equality, $S_{\text{obs}}^{AD} = \min(S_{\text{obs}}^{AB}, S_{\text{obs}}^{CD})$. This gives the hint that one can employ such an inequality as a bound for the distinguishability between a perfect bell state measurement and the black box bell state measurement.

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BLACK BOX STATE ESTIMATION

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