QUANTUM CORRELATIONS IN COMPOSITE PARTICLES

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Declarations

I hereby Declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This Thesis has also not been submitted for any degree in any university previously.

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Abstract

This thesis considers the topic of quantum correlations in the context of composite particles - larger particles that are themselves composed of more elementary bosons and fermions. The primary focus is on systems of 2 elementary fermions of a different species, a prime example of which is the hydrogen atom, although composite particles of other types are also touched upon. It turns out such systems can be made to exhibit bosonic or fermionic behaviour depending on how strongly correlated they are, as measured by the amount of entanglement these fermion pairs contain. A demonstration of how such quantum correlations in composite particles is presented, followed by explorations into their limitations and interpretation. Proposals to measure the level of bosonic and fermionic behaviours are also discussed, and their connections to work extraction in a hypothetical Quantum Szilard Engine is also studied.

Keywords:

Entanglement, Bosons, Fermions, Composite Particles

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Contents

List of Figures

This thesis is based primarily on the contents of the following papers:

- Ramanathan, R., Kurzynski, P., Chuan, T. K., Santos, M. F., Kaszlikowski, D. (2011). Criteria for two distinguishable fermions to form a boson. Physical Review A, 84(3), 034304.
- Kurzynski, P., Ramanathan, R., Soeda, A., Chuan, T. K., Kaszlikowski, D. (2012). Particle addition and subtraction channels and the behavior of composite particles. New Journal of Physics, 14(9), 093047.
- Chuan, T. K., Kaszlikowski, D. (2013). Composite Particles and the Szilard Engine. arXiv preprint arXiv:1308.1525.

Other papers not included as part of this thesis:

- Chuan, T. K., Maillard, J., Modi, K., Paterek, T., Paternostro, M., Piani, M. (2012). Quantum discord bounds the amount of distributed entanglement. Physical review letters, 109(7), 070501.
- Chuan, T. K., Paterek, T. (2013). Quantum correlations in random access codes with restricted shared randomness. arXiv preprint arXiv:1308.0476.

Chapter 1

Introduction

1.1 Elementary Particles

Most of the particles that we deal with every day are composite in nature. Members of the Periodic Table of Elements, for instance, are not actually strictly *elemental*, at least in the precise definition of the word. The table of elements actually lists *atoms*, all of which are actually composed of yet smaller, even more elemental particles, and so are composite in nature. Molecules are composed of atoms, and so are composites of composites. The everyday objects we deal with are in turn composed of molecules, and so are composite in nature as well. The properties of composite systems therefore form an important aspect of our reality. It is with this motivation that we are interested to study composite particles. A question that may then be asked is what this composite nature of particles actually adds to the physics of the system. In this thesis, we adopt what is the following perspective of the issue: the introduction of smaller constituents necessarily introduces quantum *correlations* between them, and that an understanding of these quantum correlations is necessary to understand composite particles. We will be dealing primarily with pure states, for which there is only one type of correlation we need to consider: Entanglement.

The original contributions to the subject present in this thesis are as follows: (1) A previously discovered inequality relating entanglement and the bosonization of fermion pairs is strengthened. (2) The role of entanglement as a *resource* in the boson condensation of fermion pairs is clarified. (3) A method of measuring the boson-ness and fermion-ness of composite particles is introduced, and through this measure, insights into the precise role that entanglement plays in composite particles is obtained. (4) The relationship between the amount of entanglement and its effect on work extraction in a Szilard engine is explored. The topics are discussed in a way that is intended to be as self contained as possible, although there is occasionally the odd theorem that is referenced without proof for the sake of clarity and readability.

Before introducing the topic of composite particles proper, it is worthwhile to first introduce the basic objects that these particles are made of - Bosons and Fermions.

1.1.1 Bosons and Fermions

In this section, we will briefly discuss why a broad classification of all elementary particles in nature under two umbrellas is necessary. Consider the simple case of 2 identical particles. They are identical in the strict sense, in that they are completely *indistinguishable* using any and all possible methods that is conceived or can be conceivable. We further assume that for a single particle, there is a complete set of orthogonal states which we label by the quantum number (also alternatively referred to as the mode) $m = 0, 1, 2 \dots$ and the corresponding quantum state is denoted by $|m\rangle$ in the usual Dirac notation. The description of the quantum state of a 2 particle system is then some superposition of $|m\rangle_a \otimes |n\rangle_b$ where the subscripts a and b are the particle labels. For notational simplicity, we will drop the subscripts and let the order of the quantum numbers listed dictate the particle labels, unless otherwise stated. In the 2 particle case, this means that $|m, n\rangle \equiv |m\rangle \otimes |n\rangle \equiv |m\rangle_a \otimes |n\rangle_b$.

A short argument then follows that if we accept the premise that two particles are indeed indistinguishable, then we cannot allow for every possible superposition of $|m, n\rangle$ as a valid descriptor of the system. Consider for instance the state $|m, n\rangle$ where $m \neq n$ – if a measurement of the quantum numbers has the outcome m, then it must be particle a and if the outcome is n then it must be particle b, thus allowing both particles to be distinguished from each other. Crucially, if such states are allowed, then there exists a measurement that will tell the difference when the two particles have been swapped with each other, since $|\langle m,n|n,m\rangle|^2 = 0$. This cannot be possible, since it contradicts the basic premise of indistinguishability. As such, we are forced to conclude that in the description of indistinguishable particles, not every superposition of $|m,n\rangle$ are allowed.

The opposite side of the coin are then the states which do indeed allow for a proper description of indistinguishable particles. States which do not contradict the premise of indistinguishability are the following:

$$|m,m\rangle, \tag{1.1}$$

$$(|m,n\rangle + |n,m\rangle)/\sqrt{2}\,(m \neq n), \tag{1.2}$$

as well as

$$(|m,n\rangle - |n,m\rangle)/\sqrt{2}. \tag{1.3}$$

There exists a key distinction between states of the type 1.1and 1.2 and those of type 1.3. States of type 1.1 and 1.2 are invariant when the quantum numbers of the particles are swapped with each other, and are referred to as symmetric states. On the other hand, states of the type 1.3 changes sign when the quantum numbers of the particles are swapped, and are referred to as anti-symmetric states. The important thing to note here is that for these two classes of states, the same basic physical state (up to an overall phase factor) is preserved under a swap of particle labels, thus ensuring indistinguishability. To further illustrate the point, consider the quantum state $(|m,n\rangle + e^{i\phi}|n,m\rangle)/\sqrt{2}$, where $0 \leq \phi < 2\pi$ and $n \neq m$. Swapping the particle labels, we obtain the state $(|m,n\rangle + e^{-i\phi}|n,m\rangle)/\sqrt{2}$, up to an overall phase factor. The inner product former and the latter is not equals to 1, and hence they are not identical states, unless $phi = 0, \pi$, in which case the state is either symmetric, or anti-symmetric.

It turns out that symmetric and anti-symmetric states introduce a fundamental bifurcation in the classification of particle species. The reason is because of the superposition principle. It is a basic tenet of quantum mechanics that any two valid quantum states can be superposed together to form a new equally valid state of the system under consideration. Suppose for some system of 2 indistinguishable particles, both symmetric and anti symmetric states are equally valid in describing the state of the system. This means both states $|\psi\rangle = (|m,n\rangle + |n,m\rangle)/\sqrt{2}$ and $|\phi\rangle = (|m,n\rangle - |n,m\rangle)/\sqrt{2}$ are valid since they are symmetric and anti-symmetric respectively. The equal superposition of both however, is *neither*:

$$(|\psi\rangle + |\phi\rangle)/\sqrt{2} = |m,n\rangle. \tag{1.4}$$

We have already established that the above state is not valid for indistinguishable particles. In order to preserve the principle of superposition, it will appear that the lesser evil is to postulate that systems that allow for both symmetric and anti-symmetric states simply do not exist. Indistinguishable particles are therefore either only describable by symmetric states or anti-symmetric states and not both. Particles whose states are symmetric are conventionally referred to as bosons, while particles whose states are anti-symmetric are referred to as fermions, and they are said to obey Bose-Einstein statistics, and Fermi-Dirac statistics respectively. Fortunately, this classification of particles into 2 species is validated by Nature – every known elementary particle is either a boson or a fermion.

In the case of fermions, the anti-symmetric property also leads to a peculiar property. Consider Eq. (1.3) once again, except this time, we let $m = n$. We then have:

$$(|m,m\rangle - |m,m\rangle)/\sqrt{2} = 0. \tag{1.5}$$

The result is the null vector, which is interpreted to imply that such a state cannot exist. This implies that fermions cannot have both particles sharing the exact same state! This property, which is unique to fermions, is conventionally referred to as Pauli's Exclusion Principle, or simply Pauli's Exclusion.

The complete treatment of symmetric and anti-symmetric states for systems of more than 2 particles can be found in typical quantum mechanics textbooks (See for example [1]).

In the subsequent sections however, we will employ a different, more efficient notation which will allow us to describe any number of particles. It is called the Fock Space formalism.

1.1.2 Fock Space

The use of Fock Space will allow us to efficiently treat systems of many particles, as well as to treat systems where the total number of particles may not be conserved. Examples of such systems include excited atoms that emit a photon with a certain probability, or a non-ideal optical cavity where photons my leak out into the environment. The following formalism enables us to describe such systems efficiently, and is inspired by the creation and annihilation operators developed to describe the quantum mechanical simple harmonic oscillator.

We first consider a system of consisting of only one particle species. The formalism will support the description of multiple particle species at the same time, but that generalization is relatively straightforward once the basic rules are established. Keep in mind that we are seeking a quantum mechanical description of a system with potentially any number of particles, so there must be a state representing a system with exactly zero of the particle under consideration. We call this the *vacuum* state, and represent it by $|0\rangle$. It is normalized, so $\langle 0|0\rangle = 1$.

Much like the simple harmonic oscillator, we define the creation operator $a_m^\dagger$ and call its adjoint a_m the annihilation operator. Do note that at this stage, there is no harmonic oscillator involved in the process. We simply define operators in a very *ad hoc* manner, and we will use them to describe the quantum mechanical system that we are interested in.

We then define the normalized quantum state of a single particle to be $|1_m\rangle \equiv a_m^\dagger|0\rangle$. For this one particle state, we can already see that the annihilation operator will remove a particle from the system, since

$$\langle 0|a_m a_m^\dagger|0\rangle = \langle 1|1\rangle = \langle 0|0\rangle = 1. \tag{1.6}$$

As such, $a_m a_m^\dagger|0\rangle$ must necessarily be $|0\rangle$. This is of course no coincidence and the operators are designed in such a way so as to behave like this. The operators in the quantum harmonic oscillator also behaves similarly. From this point, we will interpret the creation operator as the operator increasing the total particle number by 1, and the annihilation operator will remove a single particle from the system.

Operators Describing Fermions

Fermions have different properties from bosons, and as such, their creation and annihilation operators have to be imbued with the necessary properties that will reflect this. We know, for instance, that not more than one fermion can occupy the same state (See Eq. (1.5)). This suggests

$$(a_m^\dagger)^2 = 0, \tag{1.7}$$

which simply means that adding 2 fermions in the same state is impossible, regardless of what state it is operating on. Furthermore, Pauli's Exclusion does not apply only to the basis state since the choice of basis is completely arbitrary. This implies that even superpositions of creation operators cannot be applied twice. We consider the superposition $(a_m^\dagger + a_n^\dagger)/\sqrt{2}$:

$$\begin{aligned} \left[(a_m^\dagger + a_n^\dagger)/\sqrt{2}\right]^2 &= \frac{1}{2}a_m^\dagger a_n^\dagger + a_n^\dagger a_m^\dagger && (1.8)\\ &= \frac{1}{2}\{a_m^\dagger, a_n^\dagger\} && (1.9)\\ &= 0, && (1.10) \end{aligned}$$

where the curly braces above in Eqn (1.9) refer to the anti-commutator $\{A, B\} \equiv AB + BA$. The anti-commutative property $\{a_m^\dagger, a_n^\dagger\} = 0$ of the creation operators (and hence also the annihilation operators) is a reflection of Pauli's Exclusion Principle.

Finally, we consider another important algebraic property, the anti-commutator between the creation and annihilation operator $\{a_m, a_n^\dagger\}$. For the case where $m \neq n$, we consider the effect of the anti-commutator on an arbitrary state $|\psi\rangle$. It is clear that the only states we actually need to consider are the ones where there is no particle with the quantum number n and one particle with the quantum number m. If it were otherwise the result will always lead to the null vector, since the annihilation operator a_m cannot remove a particle occupying mode m unless one is already there, and also because the creation operator $a_n^\dagger$ cannot add a particle occupying mode n unless that mode is unoccupied due to Pauli's Exclusion. As such, we simply need to verify that

$$(a_m a_n^\dagger + a_n^\dagger a_m) a_m^\dagger |0\rangle = 0, \tag{1.11}$$

which is sufficient to prove that the anti-commutator $\{a_m, a_n^\dagger\} = 0$ for $m \neq n$. A similar check performed for the case where $m = n$ will prove that $\{a_m, a_m^\dagger\} = \nVdash$. This gives us the following general relation for fermions:

$$\{a_m, a_n^\dagger\} = \delta_{m,n}. \tag{1.12}$$

Operators Describing Bosons

Pauli's Exclusion principle does not apply for bosons, so an arbitrary number of particles can occupy each available mode. As such, the creation and annihilation operators for bosons behave most similarly to the operators of the quantum harmonic oscillator, which similarly allows for any number of excitations.

We first define the creation operators for bosons, $b_m^\dagger$ such that they obey the correct symmetric property. Bosons are symmetric under the permutation of particles. This implies that the end result of adding a particle in mode m first followed by a particle in mode n must be the same if the particles are swapped around and n was added before m. This suggests that the relevant creation operators must *commute*:

$$b_m^\dagger b_n^\dagger - b_n^\dagger b_m^\dagger = [b_m^\dagger, b_n^\dagger] = 0, \tag{1.13}$$

where the commutator is defined by $[A, B] = AB - BA$. In analogy with the number operator from the quantum harmonic oscillator, we *define* $N_m = b_m^\dagger b_m$ to be the number operator, satisfying the following property:

$$N_m|\ldots\nu_m\ldots\rangle = \nu_m|\ldots\nu_m\ldots\rangle, \tag{1.14}$$

where $|\ldots\nu_m\ldots\rangle$ is an arbitrary state with ν_m number of bosons occupying mode m. As a consequence of this requirement, we have the following expression:

$$\begin{aligned}\langle\ldots\nu_m\ldots|N_m|\ldots\nu_m\ldots\rangle &= (\langle\ldots\nu_m\ldots|b_m^\dagger)(b_m|\ldots\nu_m\ldots\rangle) && (1.15)\\ &= \nu_m. && (1.16)\end{aligned}$$

This necessarily implies

$$b_m|\ldots\nu_m\ldots\rangle = \sqrt{\nu_m}|\ldots(\nu_m-1)\ldots\rangle. \tag{1.17}$$

Since $b_m^\dagger b_m$ must be the number operator, this can be consistent with Eqn. (1.17) only if

$$b_m^\dagger|\ldots\nu_m\ldots\rangle = \sqrt{\nu_m+1}|\ldots(\nu_m+1)\ldots\rangle. \tag{1.18}$$

Finally, Eqns. (1.17) and (1.18) imply the following commutation relation for the bosonic creation and annihilation operators:

$$[b_m, b_n^\dagger] = \delta_{m,n}. \tag{1.19}$$

From this, we can tell that the key differentiator between the operators of fermions and bosons is that one obeys *anti-commutation* relations whilst the other obeys *commutation* relations, and this difference is sufficient to capture the respective peculiarities of the respective particle species. With that in mind, we now move on to the main subject matter of this thesis – composite particles.

1.1.3 Composite Particles

The study of composite particles belong to the field of many body theories. There is a large amount of literature on the subject, and it is unfortunate that the complexity of the problem usually quickly escalates as the number of particles in the system increases. There are many ways to approach the problem, and a popular approach to deal with systems of many composite particles is to launch a program of *bosonization.* The term bosonization may be used in various different contexts, but here it specifically means a systematic transformation of a problem that deal with composite particles into a problem that involves only elementary bosons, a simplification which otherwise makes intractable problems solvable. This may be physically motivated by the Spin Statistics Theorem from relativistic quantum mechanics, from which we know that bosons have integer spins and fermions have half integer spins. A pair of strongly correlated fermions would outwardly appear to have integer spin, so long as its internal structure is not probed, and is therefore expected to exhibit boson-like behaviour. For this reason, such systems are also sometimes conventionally called composite bosons, though the term is slightly misleading as not all composite systems of 2 fermions will necessarily exhibit bosonic behaviour. For more on this subject, see ([2–4]). In this thesis however, we will not be concerned with the explicit solution to many body problems. We are primarily interested in the study of how correlations present in composite particles are responsible for various physical properties of the system. As such, it is necessary for us to retain the "compositeness" of our composite particles, because it only makes sense to speak of correlations within a particle when you can subdivide said particle into partitions.

In the subsequent sections, we will primarily be dealing with systems of 2 correlated fermions and/or bosons. There are several reasons for this. One was mentioned in the previous paragraph – the structure of composite particles quickly escalate as the number of particles increases. This makes it difficult to say anything general with regards to the correlations between the particles, so only the simplest of composite systems will be studied. Another reason is that quantum correlation is very well defined in the context of 2 correlated parties. The issue becomes much more controversial as the number of parties increase beyond 2 and this is very much still an open area of research. Considering only systems of 2 correlated fermions will make the issue of correlations something that is more easily quantifiable, a quality that will be exploited, once again, in the subsequent sections.

The Operators Describing Composite Particles Of 2 Fermions

Just as we had creation operators and annihilation operators describing systems of elementary particles, we will approach composite particles in a similar manner and begin by writing down the creation operator of a composite particle. Since the algebraic properties of the respective operators encode the behaviour of the elementary particles, we can expect that the same applies to composite particles. Consider a system of 2 correlated fermions/bosons of a different type. Suppose the fermion/boson of types a and b each have a complete set of basis states which are labelled by the quantum numbers m, n respectively. The quantum state of a single composite particle looks like the following:

$$|\psi\rangle = \sum_{m,n} \lambda_{m,n} |m\rangle_a |n\rangle_b, \tag{1.20}$$

where $\sum |\lambda_{m,n}|^2 = 1$. The creation operator of a composite particle then follows straightforwardly. If $a_n^\dagger$ creates a fermion of type a in mode m, and $b^\dagger$ creates a particle of type b in mode n, then Eqn. (1.20) can be written as:

$$\begin{aligned} |\psi\rangle &= \sum_{m,n} \lambda_{m,n} a_m^\dagger b_n^\dagger |0\rangle & (1.21) \\ &= c^\dagger |0\rangle, & (1.22) \end{aligned}$$

where $c^\dagger \equiv \sum_{m,n} \lambda_{m,n} a_m^\dagger b_n^\dagger$ is the creation operator of the composite particle. From this, one may try to do something similar to elementary bosons and fermions and attempt to formulate the commutator (or anti-commutator or that matter) relations for $c^\dagger$. The intuition is that a system of 2 fermions or bosons should exhibit bosonic behaviour, since altogether they have integer spins, so one may try to evaluate $[c, c^\dagger]$. Unfortunately, the end result is typically a rather complex algebraic structure in comparison to elementary bosons. Instead of the identity, one gets:

$$[c, c^\dagger] = \mathbb{1} - \Delta, \tag{1.23}$$

where Δ is typically referred to as the "deviation from boson" operator. If Δ is close to the zero operator, then the composite particle is expected to behave like a boson as the commutation relation appears similar. Full specification of this operator will require the calculation of many matrix elements, and there exists machinery to aid this process as well as analyses of this matrix ([5–8]).

However, instead of doing that, we will adopt the approach first considered by C.K. Law [9]. We first observe that Eqn.(1.20) can afford some simplification by using the *singular value decomposition* of the matrix $\lambda_{m,n}$. In general, any matrix A can always be decomposed in the following manner ([10]):

$$A = UDV, \tag{1.24}$$

where D is diagonal and has non-negative matrix elements, and both U and V are unitary matrices. Applying this to $\lambda_{m,n}$, we have:

$$\begin{aligned}
|\psi\rangle &= \sum_{m,n} \lambda_{m,n} |m\rangle_a |n\rangle_b && (1.25)\\
&= \sum_{m,n} \sum_{k} U_{m,k} D_{k,k} V_{k,n} |m\rangle_a |n\rangle_b && (1.26)\\
&= \sum_{k} D_{k,k} \left(\sum_{m} U_{m,k} |m\rangle_a \right) \left(\sum_{n} V_{k,n} |n\rangle_b \right) && (1.27)\\
&= \sum_{k} D_{k,k} |k\rangle_a |k\rangle_b, && (1.28)
\end{aligned}$$

where$|k\rangle_a$ and $|k\rangle_b$ are defined to be $\sum_m U_{m,k}|m\rangle_a$ and $\sum_n V_{k,n}|n\rangle_b$ respectively. Note that $\langle k|k'\rangle = \delta_{k,k'}$ since U and V are unitary matrices. The above is sometimes called the *Schmidt Decomposition* of the quantum state. From this, it follows that the corresponding creation operator for the composite boson is greatly simplified so long as the correct bases sets are chosen ($|k\rangle_a$ and $|k\rangle_b$ form a complete basis for their respective particle). Without any loss in generality, we can now always assume that the creation operator of the composite particle takes the form:

$$c^\dagger = \sum_m \lambda_m a_m^\dagger b_m^\dagger. \tag{1.29}$$

This greatly simplifies our analysis. With this, we can form states of N composite particles by simply applying the creation operator N times:

$$|N\rangle = \frac{1}{\sqrt{(\chi_N N!)}}(c^\dagger)^N|0\rangle. \tag{1.30}$$

The factor χ_N is necessary in order to normalize the state so that $\langle N|N\rangle = 1$. It will soon prove important, and is defined as:

$$\chi_N \equiv \|(c^\dagger)^N|0\rangle\|^2/N!. \tag{1.31}$$

One may expect that the corresponding annihilation operator is simply the adjoint. However, this is not the case. Even though c removes 2 fermions from the system, what you get is a state that is not necessarily parallel to $|N-1\rangle$. Therefore in general:

$$c|N\rangle = \alpha_N\sqrt{N}|N-1\rangle + |\epsilon_N\rangle, \tag{1.32}$$

where $|\epsilon_N\rangle$ is simply the component of the vector is that orthogonal to $|N-1\rangle$ and α_N is some constant that is yet to be determined.

We first evaluate α_N by multiplying $\langle N-1|$ to the left of Eqn. (1.32):

$$\begin{aligned}
\langle N-1|c|N\rangle &= \alpha_N\sqrt{N} && (1.33)\\
&= \frac{|(c^\dagger)^N|0\rangle\|^2}{(N-1)!\sqrt{N(\chi_{N-1})(\chi_N)}} && (1.34)\\
&= \sqrt{N\frac{\chi_N}{\chi_{N-1}}}. && (1.35)
\end{aligned}$$

So we have

$$\alpha_N = \sqrt{\frac{\chi_N}{\chi_{N-1}}}. \tag{1.36}$$

For the next step, we need to evaluate $\langle \epsilon_N | \epsilon_N \rangle$. With α_N found, this is relatively straightforward, and one simply needs to evaluate:

$$\begin{aligned} \langle \epsilon_N | \epsilon_N \rangle &= \| c|N\rangle - \alpha_N \sqrt{N} |N-1\rangle \|^2 && (1.37) \\ &= 1 - N\alpha_N^2 + (N-1)\alpha_{N+1}^2. && (1.38) \end{aligned}$$

From Eqns. (1.36 and 1.37) it now becomes apparent that bosonic behaviour may be boiled down to a dependency on a rather simple quantity. We see that so long as $\chi_{(N\pm 1)}/\chi_N \to 1$ we will have

$$\begin{aligned} \alpha_N &\to 1 && (1.39) \\ \langle \epsilon_N | \epsilon_N \rangle &\to 0, && (1.40) \end{aligned}$$

and the ladder structure of the ideal bosonic creation and annihilation operators will be retrieved by a composite boson. At this point, we have a quantity which characterizes bosonic behaviour in a composite particle made up of 2 distinguishable fermions, but it remains unclear how to interpret it. It turns out that the factor χ_N/χ_{N-1} is related to the strength of the quantum correlation in the composite particle, which leads us to our next point of discussion.

1.2 Entanglement

Interest in the so-called *entangled* states began because of the so-called Einsten-Podolsky-Rosen (EPR) thought experiment. In their seminal paper [11], Einstein *et al.* argued that quantum mechanics must be incomplete in an ingenious argument incorporating both quantum mechanics and special relativity, and they do so by exploiting unique properties of an "EPR pair", which nowadays we call entangled states. At the time, Einstein, Podolsky and Rosen were trying to argue for the existence of an objective reality, which Quantum Mechanics with its intrinsic indeterminism appear to contradict. Unfortunately, though their physics was sound, their ultimate interpretation was not. Subsequent developments on the topic has since ruled out the possibility of any deterministic theory of the type that Einstein originally conceived. However, even though the conclusion of the paper is now largely invalidated, it does raise the possibility that entangled particles are somehow special in Quantum Mechanics. Schrödinger recognized this himself as early as 1935 after the EPR result, commenting on the EPR pairs that: "Best possible knowledge of a whole does not include best possible knowledge of its parts – and this is what keeps coming back to haunt us."

Even though entangled states have been recognised since the early days of Quantum Mechanics, our understanding of entanglement today is very much different from what Einstein and his contemporaries had in mind. Much of present day entanglement theory is spurred by discoveries in the 1990s which exploited the strangeness of entanglement in a variety of applications which include quantum cryptography [12], quantum dense coding [13] and quantum teleportation [14]. Such discoveries, all of which are experimentally demonstrated, not only revived interest in the subject but also strongly implied that entanglement constitutes a resource for which there is no classical substitute. Entangled correlations are therefore purely quantum correlations. An important development in the subsequent treatment of the subject is that entanglement, at least in the case of 2 parties, can be quantified.

We begin by defining entangled states. Suppose we have a composite Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ where $|m\rangle_a$ and $|m\rangle_b$ for $m = 0, 1$ form a complete basis for their respective Hilbert spaces. Consider the following quantum state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b). \tag{1.41}$$

The interesting thing about this state is that even though the choice of basis sets are completely arbitrary, no choice of a *local* basis of $\mathcal{H}_a$ and $\mathcal{H}_b$ will enable you to write the state as the product state $|m'\rangle_a|m'\rangle_b$ for any $|m'\rangle_a$ and $|m'\rangle_b$. It is easy to verify this by performing a calculation, but one way to see this is that both $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b)$ and $|m'\rangle_a|m'\rangle_b$ already has the form of their respective Schmidt decompositions, but one has two terms and the other only one. This implies that the two must be completely different states.

An example of a state that *can* be rewritten as a product state by a choice of bases is the following:

$$
\begin{aligned}
|\phi\rangle &= \frac{1}{2}(|0\rangle_a|0\rangle_b + |0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b) && (1.42)\\
&= \frac{1}{\sqrt{2}}(|0\rangle_a + |1\rangle_a)\frac{1}{\sqrt{2}}(|0\rangle_b + |1\rangle_b) && (1.43)\\
&= |+\rangle_a|+\rangle_b, && (1.44)
\end{aligned}
$$

where $|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Such a state we call *separable*.

Therefore, an entangled state is defined by what it is not – a pure state that cannot be written as an separable state is by definition an entangled state. Note that the above discussion involves only pure quantum states, but a more general system may be a stochastic mixture of pure quantum states. The definition of entanglement may be extended to mixed states, but for the most part, this thesis will only contain references to entanglement within pure state systems.

1.2.1 Quantifying Entanglement

Here, we introduce an entanglement measure – a quantity that serves to quantify the amount of entanglement present within a quantum state. A full discussion of entanglement measures will go far beyond the scope of this thesis. In this section however, we will try to motivate the use of the *Entropy of Entanglement* as the preferred measure of entanglement for pure quantum states. To facilitate this, we will introduce the concepts of *entanglement distillation* and *entanglement cost*.

In approaching the issue of quantifying entanglement, it makes sense to want to try to define it in an operational manner. This is because an entanglement measure is easier to make sense of if that quantity tells you something about some procedure that you are trying to perform. For instance, the number of kilograms of flour tell you how many cakes you can make, and the number of litres of water tells you how many bottles you can fill. We try to do the same thing for entanglement by finding a procedure that is enabled by its existence, following which we can then try to quantify it.

It turns out that the key to this is a physical constraint typically referred to as the *Local Operations, Classical Communication (LOCC)* constraint. Under this constraint, Alice and Bob are in laboratories separated by some distance. They are not allowed to communicate quantum states but are allowed to communicated classical bits and perform any operation locally, quantum or otherwise. This constraint arise from the observation that it is much more difficult to communicate a quantum bit containing quantum information than a classical bit continuing a classical message. Two parties wishing to communicate quantum bits (some quantum state in a superposition of $|0\rangle$ and $|1\rangle$) can sidestep this limitation however, if they share entangled quantum bits (qubits). If two parties, Alice and Bob have entangled qubits readily available, they can perform quantum teleportation via a well defined procedure [14] that involves Alice and Bob performing only local actions and communicating classical messages to each other. This means that Alice can send qubits to Bob without physically transporting her qubits. Furthermore, if Alice and Bob starts with product states, there does not exist any locally performed procedure for them to produce entangled states! Suppose Alice and Bob shares some product state $|\psi\rangle_a|\phi\rangle_b$ and performs a local unitary operation U_a and U_b to their respective qubits:

$$\begin{aligned} U_a \otimes U_b|\psi\rangle_a|\phi\rangle_b &= (U_a|\psi\rangle_a)(U_b|\phi\rangle_b) \\ &= |\psi'\rangle_a|\phi'\rangle_b, \end{aligned} \tag{1.45}$$

which is again a product state, and thus not entangled. The same argument also applies if Alice and Bob are also allowed more general quantum operations. As such, the LOCC constraint makes entanglement a useful resource for Alice and Bob to have prior to the start of their communication. It allows them to bypass limitations in communicating quantum states.

So we now have a procedure to give our eventual entanglement measure a more physical meaning. It also makes sense for us to define a standard unit of entanglement, just like mass and distance has standard units in the form of the kilogram and metre. For this purpose, we decide to choose the following entangled state as our standard unit:

$$|\psi\rangle_{ab} = \frac{1}{\sqrt{2}}(|00\rangle_{ab} + |11\rangle_{ab}). \tag{1.46}$$

We therefore have the requirement that the entanglement measure gives this state an entanglement of 1. This state is also sometimes referred to as the Bell state. The choice of this state is natural in the context of the quantum teleportation procedure. If you have the state (1.46), Alice and Bob can perfectly communicate 1 qubit between them under the LOCC constraint.

We now consider how to quantify the amount of entanglement for some arbitrary pure state between Alice and Bob $|\phi\rangle_{ab}$. Now, if Alice and Bob is using some state $|\phi\rangle_{ab}$ as a resource for communicating quantum information, they will most likely attempt to stockpile as many pairs of the states as possible. Let the number of pairs of the state $|\phi\rangle_{ab}$ be N, which is intended to be some large integer. We now ask how many pairs of state of the type (1.46) is necessary in order to reproduce the state $|\phi\rangle_{ab}^{\otimes N}$. This may seems like an arbitrary question at this juncture, given that we have not even discussed the possibility of transforming pairs of $|\psi\rangle_{ab}$ into some other state, but it is actually quite intuitive why this is possible. It is clear, for instance, that Alice and Bob can locally prepare the state $|\phi\rangle_{ab}$ and send them to each other if they are allowed to send quantum messages. Of course, this is not allowed to do this due to the LOCC limitation, but we have previously made the association that the Bell state in Eqn. (1.46) enables the communication of 1 qubit of quantum information using only LOCC. As such, a sufficient number of pairs of the state $|\psi\rangle_{ab}$ paired with LOCC should be able to reproduce any arbitrary state. By considering the minimum number of pairs of $|\psi\rangle_{ab}$ required to produce a state, we have a reasonable count of how much entanglement there is within it, so to speak.

It turns out that this question is readily answered for the special case of pure states (for mixed states, it is a far more complex issue). We first write the quantum state $|\phi\rangle_{ab}$ in its Schmidt decomposition:

$$|\phi\rangle_{ab} = \sum_m \sqrt{p_m}|m,m\rangle_{ab}, \tag{1.47}$$

where $m = 0, 1, 2, \ldots, m_{max}$. The state of N copies of this state can be written as:

$$|\phi\rangle_{ab}^{\otimes N} = \sum_{m_1,m_2,\ldots,m_N} \sqrt{p_{m_1}p_{m_2}\cdots p_{m_N}}|m_1,m_2,\ldots,m_N\rangle_a|m_1,m_2,\ldots,m_N\rangle_b. \tag{1.48}$$

We consider the product $p_{m_1}p_{m_2}\ldots p_{m_N}$ as probabilities, and appeal to the law of large numbers. As the number N gets very large, the vast majority of the sequences are "typical" and any other sequence can be safely ignored. A typical sequence $m_1, m_2, \ldots, m_N$ contains Np_0 number of 0s, Np_1 number of 1s and so on. This is a very intuitive outcome: as the length of the random sequence N gets longer and longer, the the proportion of i in the sequence is increasing likely to be the the probability of getting i, i.e. $\frac{n_i}{N} = p_i$. All typical sequences have the same coefficients in the Eqn. (1.48). Since there are so few non-typical sequences in comparison, we can safely ignore them and write the state as:

$$|\phi\rangle_{ab}^{\otimes N} = \sum_{m_1,m_2,\ldots,m_N \text{ typical}} \frac{1}{\sqrt{K}}|m_1,m_2,\ldots,m_N\rangle_a|m_1,m_2,\ldots,m_N\rangle_b, \tag{1.49}$$

where $K = \frac{N!}{n_0!n_1!\ldots n_{max}!}$ is the total number of typical sequences. This can be simplified using Stirling's approximation $\log(N!) \approx N\log N - N$ and the end result is:

$$\begin{aligned}
\frac{\log K}{N} &\approx \sum_{i=0}^{m_{max}} -\frac{n_i}{N}\log\frac{n_i}{N} && (1.50)\\
&= \sum_{i=0}^{m_max} -p_i \log p_i && (1.51)\\
&\equiv H(p_i). && (1.52)
\end{aligned}$$

Once again, the above approximation can be considered to be an equality as N gets very large. Expression (1.51) has a special name: Shannon entropy [15, 16], and has a variety of applications in information theory. Regardless, this suggests that if Alice and Bob wishes to share the state $|\phi\rangle_{ab}^{\otimes}N$, all she needs to do is locally prepare the state $\frac{1}{\sqrt{K}}\sum_j^K |j\rangle_a|j\rangle_b$ and send half of it over to Bob. Alice needs to send over at least $\log K$ qubits to do this because that is the minimum number of qubits necessary to encode K orthogonal vectors. As a result, she needs $\log K$ pairs of our entangled state $|\psi\rangle_{ab}$ to be able to communicate those qubits using LOCC, so on average the number of Bell states required per copy of $|\phi\rangle_{ab}$ is $\frac{\log K}{N} = H(p_i)$. This quantity we call the *entanglement cost* of the pure state $|\phi\rangle_{ab}$.

It turns out that this process is reversible! You can begin with N copies of $|\phi\rangle_{ab}$ and through some LOCC process, produce, on average $\frac{\log K}{N} = H(p_i)$ copies of the Bell state per copy of $|\phi\rangle_{ab}$. We have already noted that $|\phi\rangle_{ab}^{\otimes N}$ is essentially an even superposition of K terms when N is large. It is easy to verify that $\log K$ copies of the Bell state $|\psi\rangle_{ab}$ is also an even superposition of K terms. Therefore, from $|\phi\rangle_{ab}^{\otimes N}$ all Alice and Bob needs to do is to perform a local unitary as appropriate to get $\frac{\log K}{N} = H(p_i)$ pairs of Bell states per copy of $|\phi\rangle_{ab}$. The entanglement measure from the process of creating Bell pairs from the state $|\phi\rangle_{ab}$ is referred to as *distillable entanglement*. As it turns out, from the above argument, that entanglement cost and distillable entanglement are equal for pure states, and is given by $H(p_i)$. As such, we give this quantity a unique name: *entropy of entanglement*.

Entropy of Entanglement and Related Quantities

In the previous portion, we motivated the use of the Shannon entropy $H(p_i)$ as a means to quantify entanglement, with the quantity having physical interpretations in terms of the entanglement cost, or in terms of the distillable entanglement. The entropy of entanglement of a pure state $|\phi\rangle_{ab}$ is defined to be:

$$E(|\phi\rangle_{ab}) \equiv -\mathrm{Tr}(\rho_a \log \rho_a) \tag{1.53}$$
$$\equiv -\mathrm{Tr}(\rho_b \log \rho_b), \tag{1.54}$$

where $\rho_a \equiv \mathrm{Tr}_b(|\phi\rangle_{ab}\langle\phi|)$ and $\rho_b \equiv \mathrm{Tr}_a(|\phi\rangle_{ab}\langle\phi|)$ are the reduced density matrices of the state $|\phi\rangle_{ab}$. The term $-\mathrm{Tr}\rho \log \rho$ where ρ is a density matrix representing some quantum state is also call the von Neumann entropy. It can be verified that this definition is identical to the one given previously. One simply needs to use the Schmidt form in Eqn.(1.47) to check that $E(|\phi\rangle_{ab}) = H(p_i)$, as was previously claimed. The entropy of entanglement is therefore simply the entropy of the subsystem a or subsystem b. We note that the entropy $H(p_i)$ is a measure of uncertainty, and the larger it is, the more uncertain we are about a particular system. This brings us back to the quote by Schrodinger in 1935: "Best possible knowledge of a whole does not include best possible knowledge of its parts". The entropy of entanglement captures this quintessential aspect of entanglement by saying that the more entangled the state, the less knowledge we have regarding its subsystem. This suggests that outside of the von Neumann entropy of entanglement we defined above, we may just as well use other measures of uncertainty/entropies to quantify entanglement. The only problem with this, of course, is that the Rényi entropy as an entanglement measure does not necessarily have operational significance, although the quantity by itself has applications. On the other hand, the benefit of using other entropies however is that it may allow us to make contact with physical problems where the von Neumann entropy does not naturally appear. It may also ease the computational requirements involved in quantifying entanglement in many cases. As such, we introduce the following generalization of the von Neumann entropy, the Rényi entropy of a quantum state ρ [17]:

$$H_\alpha(\rho) \equiv \frac{1}{1-\alpha} \log \mathrm{Tr}\rho^\alpha, \tag{1.55}$$

where $\alpha \leq 0$ and $\alpha \neq 1$. It can be shown (as a simple application of L'Hospital's rule) that in the limit of $\alpha \to 1$ the above definition results in the usual Shannon's entropy. It is also a non-increasing function of α, which one can verify by computing the derivative. Rényi entropies therefore form an entire continuous class of entropies. In order for us to make contact with composite particles however, we consider the case of $\alpha = 2$. The corresponding entropy is then given as:

$$H_2(\rho) = -\log \mathrm{Tr}\rho^2 = -\log P. \tag{1.56}$$

The quantity $P \equiv \mathrm{Tr}\rho^2$ where $0 < P \leq 1$ is frequently referred to as purity, a measure of how pure a given quantum state is. Since $\alpha > 1$,if one wishes to make a connection with the von Neumann entropy $H(\rho)$, then the Eqn. (1.56) is a lower bound, i.e. $H(\rho) \geq H_2(\rho)$. The quantity $H_2(\rho)$ may then be used to quantify entanglement of a bipartite system, but since the quantity itself has no operational significance at this point, we can make a further simplification. Observe that $H_2(\rho) = -\log P$ is a strictly monotonically decreasing function of P. Another, very simple monotonically decreasing function of P is $1-P$, which has a direct 1 to 1 correspondence with the function $H_2(\rho)$. The quantity $1-P$, when applied to the subsystem of pure bipartite state, is therefore also a reasonable measure of entanglement. This quantity is also given a special name: the linear entropy of entanglement.

The Entanglement In Composite Particles

In this section, we elaborate upon the relationship between entanglement, as discussed previously, and their relation composite particles. The relationship between entanglement and composite particles is first rigorously proven in [18] through the following inequality.

$$1 - NP \leq \frac{\chi_{N+1}}{\chi_N} \leq 1 - P, \tag{1.57}$$

where the P in this case is the purity of the particle a for a *single* composite boson(or b, since they are the same). We recall from Section (1.1.3) that the quantity $\frac{\chi_{N+1}}{\chi_N}$ measures how "bosonic" the creation and annihilation operator of the composite particle is. The closer it is to 1, the more bosonic the composite particle is since the creation and annihilation operators start to behave like ideal boson creation and annihilation operators. As argued in the previous section, the quantity $1 - P$ is an entanglement measure, therefore as the entanglement approaches 1 and $P \to 0$, the above inequality suggests that $\frac{\chi_{N+1}}{\chi_N} \to 1$, and the composite particle approaches an ideal boson. As such, we see that for composite bosons, the amount of entanglement correlates with how bosonic it is. Subsequently, we present the proof of the inequality.

We first prove the lover bound $\chi_{N+1}/\chi_N \leq 1 - NP$. In order to achieve this, we simply need to verify that $\chi_{N+1} - \chi_N(1 - NP)$ is non negative. But we first take a look at the amount of entanglement for a single composite particle.

$$c^\dagger|0\rangle = \sum_m \sqrt{\lambda_m}|m, m\rangle_{ab}. \tag{1.58}$$

It is easy to verify that the reduced density matrix of the subsystem a is the following:

$$\rho_a = \mathrm{Tr}_b(c^\dagger|0\rangle\langle 0|c) = \sum_m \lambda_m |m\rangle_a \langle m|. \tag{1.59}$$

The purity of the above reduced density matrix is then simply

$$\mathrm{Tr}_a(\rho_a^2) = \sum_m \lambda_m^2. \tag{1.60}$$

We also require the following:

$$\begin{aligned}
\chi_N &= \langle 0|c^N (c^\dagger)^N|0\rangle/N! && (1.61)\\
&= \frac{1}{N!}(N!)^2 \sum_{m_1 < \cdots < \, m_N} \lambda_{m_1}\lambda_{m_2}\ldots\lambda_{m_N} && (1.62)\\
&= (N! \sum_{m_1 < \cdots < \, m_N} \lambda_{m_1}\lambda_{m_2}\ldots\lambda_{m_N}) && (1.63)\\
&= \sum_{m_1 \neq \cdots \neq \, m_N} \lambda_{m_1}\lambda_{m_2}\ldots\lambda_{m_N} && (1.64)\\
&= \sum_{m_1 \neq \cdots \neq \, m_N} \prod_{j=1}^{N} \lambda_{m_j}. && (1.65)
\end{aligned}$$

It is interesting to note that sums of the type above are well studied and and are called *elementary symmetric functions*. Regardless, all that is necessary to prove the required inequality is some algebra. Substituting Eqns. (1.60) and (1.61) below:

$$\begin{aligned} &\chi_{N+1} - \chi_N(1 - NP) && (1.66)\\ &= \sum_{m_1 \neq \cdots \neq\ m_{N+1}} \prod_{j=1}^{N+1} \lambda_{m_j} - (1 - N\sum_n \lambda_n^2) \sum_{m_1 \neq \cdots \neq\ m_N} \prod_{j=1}^{N} \lambda_{m_j}. && (1.67) \end{aligned}$$

Many of the terms in the above sum is identical. By systematically cancelling out the terms that subtract from each other, it is possible to verify that the remaining terms left may be written as:

$$\begin{aligned} &\chi_{N+1} - \chi_N(1 - NP) && (1.68)\\ &= N(N-1) \sum_{m_1 \neq \cdots \neq m_N} \prod_{j}^{N} \lambda_{m_j} (\lambda_{m_1}^2 - \lambda_{m_1}\lambda_{m_2}) && (1.69)\\ &= \frac{N(N-1)}{2} \sum_{m_1 \neq \cdots \neq m_N} \prod_{j=1}^{N} \lambda_{m_j} (\lambda_{m_1}^2 + \lambda_{m_2}^2 - 2\lambda_{m_1}\lambda_{m_2}) && (1.70)\\ &= \frac{N(N-1)}{2} \sum_{m_1 \neq \cdots \neq m_N} \prod_{j=1}^{N} \lambda_{m_j} (\lambda_{m_1} - \lambda_{m_2})^2 && (1.71)\\ &\geq 0, && (1.72) \end{aligned}$$

which proves the lower bound $\frac{\chi_{N+1}}{\chi_N} \geq 1 - NP$.

We may prove the upper bound in a similar manner, by proving that $(1-P)\chi_N - \chi_{N+1} \geq 0$.

$$(1-P)\chi_N - \chi_{N+1} \tag{1.73}$$
$$= (1 - \sum_{m_{N+1}} \lambda^2_{m_{N+1}}) \sum_{m_1 \neq \cdots \neq m_N} \prod_j^N \lambda_{m_j} - \sum_{m_1 \neq \cdots \neq m_{N+1}} \prod_j^{N+1} \lambda_{m_j} \tag{1.74}$$
$$= (N-1) \sum_{m_1 \neq \cdots \neq m_{N+1}} \left(\prod_j^N \lambda_{m_j} \lambda^2_{m_{N+1}} + N(N-1) \prod_j^{N-1} \lambda_{m_j} \lambda^2_{m_N} \lambda^2_{m_{N+1}} \right) \tag{1.75}$$
$$\geq 0, \tag{1.76}$$

where the above inequality is again achieved by systematically matching identical terms. Together, Eqns (1.68) and (1.73) proves the required inequality $1 - NP \leq \frac{\chi_{N+1}}{\chi_N} \leq 1 - P$, so we make contact between composite bosons and entanglement.

However, note that the above upper bound $\frac{\chi_{N+1}}{\chi_N} \leq 1 - P$ does not depend on the number of composite particles N in general, so for larger systems, the possible range of values of $\frac{\chi_{N+1}}{\chi_N}$ increases. It is possible to get around this limitation by proving a tighter upper bound. To do this, we require the *Schmidt number* of the composite particle. This is simply the number of non-zero coefficients $\sqrt{\lambda_n}$ in the state of one composite particle given in Eqn. (1.58). We will denote the Schmidt number by $m_{\max}$. In order to prove a tighter upper bound, we will require prior knowledge of an inequality involving the elementary symmetric functions $\chi_N/N!$:

$$\frac{\chi_{N+1}}{\chi_N} \leq \frac{(m_{\max} - N)}{(m_{\max} - N + 1)} \frac{\chi_N}{\chi_{N-1}}, \tag{1.77}$$

where the above inequality is known as *Newton's inequality* [19]. From the above inequality it is easy to see, for instance that the function $\frac{\chi_{N+1}}{\chi_N}$ is a non-increasing function of N. We are primarily interested in its relation to the purity P however. By applying the above inequality repeatedly, it is possible to verify the following:

$$
\begin{aligned}
\frac{\chi_{N+1}}{\chi_N} &\leq \frac{(m_{\max}-N)(m_{\max}-N-1)\ldots(m_{\max}-2)}{(m_{\max}-N+1)(m_{\max}-N)\ldots(m_{\max}-1)}\frac{\chi_2}{\chi_1} && (1.78)\\
&= \frac{m_{\max}-N}{m_{\max}-1}\chi_2 && (1.79)\\
&= g(m_{\max},N)\chi_2, && (1.80)
\end{aligned}
$$

where $g(m_{\max},N) \equiv \frac{m_{\max}-N}{m_{\max}-1}$ is a non-increasing function of N. We then observe that $\chi_2 = \sum_{m_1,m_2}\lambda_{m_1}\lambda_{m_2} - \sum_{m_1=m_2}\lambda_{m_1}\lambda_{m_2} = 1-P$ is simply the linear entropy of entanglement, so we get the following tighter upper bound:

$$
\frac{\chi_{N+1}}{\chi_N} \leq g(m_{\max},N)(1-P). \qquad (1.81)
$$

1.3 Conclusion

In this chapter, we briefly introduced the relevant concepts and notion that will be used in the subsequent chapters.

We began first by introducing the concepts of bosons and fermions, and how they may each be described using the Fock space formalism through operators that capture their essential properties. We go on to apply this concepts by describing operators of composite particles made up of 2 distinguishable particles, either bosons or fermions.

Chapter 2

Entanglement and its Relation to Boson Condensation

In the previous chapter, we briefly introduced how to measure pure state entanglement, and how entanglement is intricately related to the way that pairs of fermions may exhibit interesting bosonic behaviour. Principle to our discussion is that a system of fermion pairs do not automatically assume bosonic behaviour, and thus the bosonic nature of such pairs cannot be taken for granted. Knowledge of a single quantity (the amount of entanglement within a single composite particle), however, appear to characterize this "bosonification" of fermion pairs well.

In this chapter, we are primarily interested in the limits of entanglement in characterizing bosonic behaviour. We begin by considering a scenario where fermion pairs *do* exhibit a clear bosonic effect, and study to what extent entanglement is responsible for it. A well known natural phenomena that is attributed to the boson-like properties of systems of fermions is the *Bose-Einstein condensate (BEC).* In the context of a BEC, composite fermions such as excitons (See for example [20]) have been studied in detail and it was suggested that particle densities and wave function overlap is necessary for a BEC to occur. It is therefore interesting to ask if knowledge of the entanglement in a composite particle is also sufficient for us to predict whether a BEC is in principle possible. That is, we would like to know if the amount of entanglement encodes sufficient information regarding the system that will allow us to say for certain whether a BEC may occur.

2.1 State Transformations Utilizing Entanglement

Typical treatments of a BEC usually require some consideration of statistical thermodynamics. We will not be tackling these complexities, and will instead consider a different approach to the problem. The key defining feature of a BEC is the mass occupation of a single quantum state, typically the ground state of the boson. It is natural to ask whether entanglement is responsible for this effect. Here, we ask the question: does the entanglement of a composite particle enable the mass occupation of N composite particles in a quantum state? Before we can answer this question, it is necessary for us to introduce a tool that relates entanglement to state transformations. The rest of this section will therefore be devoted to establishing some necessary facts regarding LOCC transformations of states.

2.1.1 LOCC Transformations of States

Consider again the LOCC paradigm (See Section 1.2.1 for an explanation). As previously discussed, entanglement may be viewed as a type of correlation that allows you to overcome the limitations of communicating quantum states under the LOCC constraint. It turns out that if Alice and Bob share some initial entangled state $|\psi\rangle_{ab}$, they can, if they so choose, cooperate and transform their state to another state $|\phi\rangle_{ab}$ so long as certain condition are satisfied. The initial state always needs to be more entangled than the final state because LOCC procedures cannot increase the amount of entanglement on average. This process is called *entanglement transformation.* The rest of this section will be primarily discussing this process.

In order to demonstrate that entanglement transformation is possible, we need a few relevant facts regarding a mathematical tool called majorization. Suppose we have a vector $x = (x_1, \ldots, x_d)$. An ordered vector $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_d^{\downarrow})$ is simply the vector x with its elements arranged in a decreasing order, i.e. $x_1^{\downarrow} \leq \cdots \leq x_d^{\downarrow}$. We say that y majorizes x or $x \prec y$ if the following inequality is satisfied:

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow} \tag{2.1}$$

for all $k = 1, \ldots, d-1$. For d, we require an equality, $\sum_{j=1}^{d} x_j^{\downarrow} = \sum_{j=1}^{d} y_j^{\downarrow}$. We can extend this definition to any 2 Hermitian operators X and Y in a similar way by constructing vectors x and y whose components are the eigenvalues of X and Y respectively. If the vectors of eigenvalues of Y majorizes X, then we also say Y majorizes X and $X \prec Y$.

It turns out that there are several alternative ways to define majorization. One way is through the following theorem:

Theorem 2.1. *$x \prec y$ if and only if $x = \sum_j p_j P_j y$ for some probability distribution p_j and permutation matrices P_j. x is therefore some convex combination of permutations of y.*

Note that the identity matrix $\mathbb{1}$ is also considered a permutation matrix. It is not immediately clear why the above statement is equivalent to the majorization condition. The proof is sufficiently elementary to be presented below:

Proof. The proof of the forward ($x \prec y$ implies $x = \sum_j p_j P_j y$)essentially works by construction. Given $x \prec y$ we will systematically construct x from y by permuting the elements of the vector. For this purpose, we will define the permutation matrix $P_{i,j}$ to be the permutation matrix with permutes the position of the ith and jth elements of the vector y. That is:

$$P_{i,j} y = P_{i,j}(\ldots, y_i, \ldots, y_j, \ldots) = (\ldots, y_j, \ldots, y_i, \ldots). \tag{2.2}$$

We will also assume that the elements of x and y are already arranged in decreasing order. There is no loss in generality with this assumption, since if it were otherwise, the ordered versions are just yet another layer of permutations away from the unordered version. We begin constructing the vector x by recreating the first element x_1. Since $x \prec y$, we have $x_1 \leq y_1$. In order to get x_1 through a convex combination, we need to find the first element in the ordered vector y such that $y_j \leq x_1$. Note that this means every element before y_j, $y_1, \ldots, y_{j-1}$ is greater or equal to x_1.

As $y_j \leq x_1 \leq y_1$, we can always write the convex combination $x_1 = py_1 + (1-p)y_j$. We can therefore get the first element of x correct through the following convex combination:

$$p\nVdash y + (1-p)P_{1,j}y \quad = \quad \begin{pmatrix} p(1)y_1 + (1-p)y_j \\ y_2 \\ \vdots \\ y_{j-1} \\ p\nVdash y_j + (1-p)y_1 \\ y_{j+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \\ \vdots \\ y_{j-1} \\ p\nVdash y_j + (1-p)y_1 \\ y_{j+1} \\ \vdots \end{pmatrix}. \tag{2.3}$$

This allows the construction of the first element x_1 through a convex combination of permutations of y. Note that the vector on the right $y' \equiv p\nVdash y + (1-p)P_{1,j}y$ still majorizes x, so $x \prec y'$. This is easy to verify. For any integer $k < j$, $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y'_i$ since by definition j is the first element smaller than x_1, so $k < j$ imply $y_k \geq x1 \geq x_k$. For $k \geq j$, the sum of the first k elements of y' is the same as the sum of the first k terms of y, i.e. $\sum_{i=1}^{k} y'_1 = \sum_{i=1}^{k} y_1$, so the rest of the majorization conditions are obeyed. In addition, note that the first element of x and y' are identical, so we have for $k \geq 2$:

$$\sum_{i=1}^{k} x_i \quad \leq \quad \sum_{i=1}^{k} y'_i \tag{2.4}$$

$$x_1 + \sum_{i=2}^{k} x_i \quad \leq \quad x_1 + \sum_{i=2}^{k} y'_i \tag{2.5}$$

$$\sum_{i=2}^{k} x_i \quad \leq \quad \sum_{i=2}^{k} y'_i. \tag{2.6}$$

This means that even if we remove the first element of x and y', the resulting vector remains majorized in the same direction. Since a similar inequality is all that was necessary for us to construct x_1 through a convex combination of permutations of y, reiterate the previous process for x_2 and construct it via a convex combination of permutations of

y'. This can be repeated for all subsequent terms until x_d, the last element of x, is also reconstructed. This is sufficient to establish that x can always be written as a convex sum of permutations of y, and establishes the forward direction of Thm. (2.1).

To establish the reverse direction (i.e. $x = \sum_j p_j P_j y$ implies $x \prec y$), note y is ordered in decreasing order so the identity permutation always leads to the largest sum of the first k elements. This establishes that $\sum_i^k x_i = \sum_i^k (\sum_j p_j P_j y)_i \leq \sum_i^k y_i$. In addition, $\sum_i^d x_i = \sum_i^k y_i$ since $\sum_i^d (P_j y)_i = \sum_i^d y_i$. This concludes the proof. □

Another equivalent definition is obtained through the following theorem:

Theorem 2.2. *$x \prec y$ if and only if $x = Dy$ where D is a doubly stochastic matrix which satisfies $\sum_i D_{ij} = \sum_j D_{ij} = 1$.*

The above is a direct consequence of *Birkhoff's Theorem* [21], which essentially states that every doubly stochastic matrix is a convex combination of permutation matrices, i.e. $D = \sum_i p_i P_i$ where p_i defines a probability distribution and P_i is a permutation matrix.

Relevant to our interests is how these definitions apply to density matrices. For this, we have to establish the following important fact:

Theorem 2.3. *Let ρ and ρ' be density operators. Then $\rho \prec \rho'$ if and only if $\rho = \sum_j p_j U \rho' U^\dagger$ for some probability distribution p_j.*

Proof. This theorem is essentially the density matrix equivalent of Thm. (2.1). We first establish that $\rho \prec \rho'$ implies $\rho = \sum_j p_j U_j \rho' U_j^\dagger$. For convenience, we define the diagonal matrices $D(\rho)$ and $D(\rho')$ to be the matrix of (ordered) eigenvalues of their respective density matrices. Since $\rho \prec \rho'$, the non-zero elements of $D(\rho)$ is some convex combination of permutations of the diagonals of matrix $D(\rho')$, i.e. $D(\rho) = \sum_j p_j P_j D(\rho') P_j^\dagger$. However, $D(\rho') = V \rho' V^\dagger$ since it is the diagonalization of ρ', so $D(\rho) = \sum_j p_j P_j V \rho' V^\dagger P_j^\dagger = \sum_j p_j U_j \rho' U_j^\dagger$ where $U_j = P_j V$. There is no loss in generality by assuming ρ is already diagonal, which concludes the proof of the forward direction.

We now show that $\rho = \sum_j p_j U \rho' U^\dagger$ implies $\rho \prec \rho'$. There is no loss in generality to assume both ρ and ρ' are already diagonal (all the unitaries can be rearranged and absorbed into U), so $D(\rho) = \sum_j p_j U_j D(\rho') U_j^\dagger$. In component form it is written

$(D(\rho))_{mm} = \sum_j p_j |U_{j,mn}|^2 (D(\rho))_{nn}$, from which it can be verified that the matrix defined by $\sum_j p_j |U_{j,mn}|^2$ is doubly stochastic as the rows and columns sum to one. This shows that $\rho \prec \rho'$ as a consequence of Thm. (2.2). □

The above theorem then allows us to demonstrate the following important fact regarding state transformations using LOCC:

Theorem 2.4. *A bipartite state $|\phi\rangle_{ab}$ may be transformed to another state $|\phi'\rangle_{ab}$ using only LOCC if and only if $\rho_a \prec \rho'_a$, where $\rho_a \equiv \mathrm{Tr}_b(|\phi\rangle_a b\langle\phi|)$ and $\rho'_a \equiv \mathrm{Tr}_b(|\phi'\rangle_a b\langle\phi'|)$.*

Proof. We begin by first noting that any local operation Bob wishes to perform on his subsystem may be achieved by Alice performing a measurement and sending her outcome to Bob, who upon receiving the message, will perform a suitable unitary. Any local operation (including measurement) Bob can perform on his subsystem is described by a set of Kraus operators $B_i = \sum_{m,n} B_{i,mn} |m\rangle_b \langle n|$ satisfying $\sum_i B_i B_i^\dagger = \mathbb{1}$. Measurement operators are also defined similarly. Applying this to a quantum state, we have:

$$B_i|\phi\rangle_{ab} = B_i \sum_m \sqrt{p_m} |m,m\rangle_{ab} \tag{2.7}$$

$$= \sum_{m,n} B_{i,nm} \sqrt{p_m} |m,n\rangle_{ab}. \tag{2.8}$$

However, Alice can perform a measurement on her side described by operators $A_i = \sum_{m,n} B_{i,mn} |m\rangle_a \langle n|$. Applying it to her subsystem leads to the following state:

$$A_i|\phi\rangle_{ab} = A_i \sum_m \sqrt{p_m} |m,m\rangle_{ab} \tag{2.9}$$

$$= \sum_{m,n} B_{i,nm} \sqrt{p_m} |n,m\rangle_{ab}. \tag{2.10}$$

Note that Eqns (2.7) and (2.9) differ only in the local basis but otherwise Alice and Bob is correlated in the same way! Therefore, any local operations on Bob's side may be substituted by an measurement operation on Alice's side, followed by appropriate local unitaries.

Based on this fact, it is now simple to prove the forward direction of the theorem. We now consider an LOCC that transforms $|\phi\rangle$ to $|\phi'\rangle$. Since the final state is a pure bipartite state and not a statistical mixture, this implies that $A_i\rho_a A_i^\dagger = p_i\rho'_a$. Any matrix M can always be written in terms of its polar decomposition such that $M = \sqrt{MM^\dagger}U$ where U is some unitary. As a consequence, we get $A_i\sqrt{\rho_a} = \sqrt{A_i\rho_a A_i^\dagger}U_i$. We can then verify the following:

$$\begin{aligned} \rho_a &= \sum_i \sqrt{\rho_a}A_i^\dagger A_i\sqrt{\rho_a} && (2.11)\\ &= \sum_i (\sqrt{A_i\rho_a A_i^\dagger}U_i)^\dagger \sqrt{A_i\rho_a A_i^\dagger}U_i && (2.12)\\ &= \sum_i (\sqrt{p_i\rho'_a}U_i)^\dagger \sqrt{p_i\rho'_a}U_i && (2.13)\\ &= \sum_i p_i U_i^\dagger \rho'_a U_i. && (2.14) \end{aligned}$$

This is sufficient to show that if the transformation from $|\phi\rangle$ to $|\phi'\rangle$ is possible using LOCC, then it implies that $\rho_a \prec \rho'_a$, as a consequence of Thm. (2.3)

We now prove the reverse direction. Given that $\rho_a \prec \rho'_a$, then it must be possible to write $\rho_a = \sum_j p_j U_j \rho'_a U_j^\dagger$ where U is a unitary operator that can be decomposed into the form $P_j V$ (See proof of Thm. (2.3)). P_j is some permutation matrix, and V is some unitary such that $V\rho'_a V^\dagger$ is diagonal. Again, we assume that ρ_a is already diagonalized without any loss in generality. Suppose the basis $\{|m\rangle_a\}$ and $\{|m'\rangle\}$ diagonalizes ρ_a and ρ'_a respectively. In particular, we have $V^\dagger|m\rangle = |m'\rangle$. From this, we define the measurement performed by Alice to be $A_i = \sqrt{p_i\rho'_a}P_i V^\dagger \rho_a^{-\frac{1}{2}}$. Suppose $|\phi\rangle_{ab} = \sum_m \sqrt{p_m}|m, m\rangle_{ab}$ and $|\phi'\rangle_{ab} = \sum_{m'} \sqrt{p'_{m'}}|m', m'\rangle_{ab}$. It is then easy to verify the following:

$$\begin{aligned}
A_i|\phi\rangle_{ab} &= \sqrt{p_i\rho'_a}P_iV^\dagger\rho_a^{-\frac{1}{2}}\sum_m\sqrt{p_m}|m,m\rangle_{ab} &(2.15)\\
&= \sqrt{p_i\rho'_a}P_i\sum_m\frac{1}{\sqrt{p_m}}\sqrt{p_m}V^\dagger|m,m\rangle_{ab} &(2.16)\\
&= \sqrt{p_i\rho'_a}P_i\sum_m|m',m\rangle_{ab} &(2.17)\\
&= \sqrt{p_i}\sum_m\sqrt{p'_{\sigma^i_{m'}}}|\sigma^i_{m'},m\rangle_{ab}, &(2.18)
\end{aligned}$$

where $\{\sigma^i_1,\ldots,\sigma^i_d\}$ is some permutation of $\{1,\ldots,d\}$ corresponding to the permutation matrix P_i. Observe that Eqn. (2.18) differs from $|\phi'\rangle$ only by a local unitary on Bob's side, which Alice can instruct Bob to perform on his subsystem via classical communication. This is sufficient to prove that if $\rho_a \prec \rho'_a$, then there exist an LOCC operation transforming $|\phi\rangle_{ab}$ to $|\phi'\rangle_{ab}$. □

2.2 Condensation Using LOCC

In the previous section, we have discussed how *majorization* is both necessary and sufficient for states to be able to transform into each other using only LOCC. An experimentalist in the laboratory trying to make a condensate however, is not limited to only using LOCC operations. In a typical setup, a very large number N number of composite particles are situated in close proximity to each other, granting ample opportunities for direct interactions between their components. Such interactions are freely able to increase the amount of entanglement within the composite particles via an interacting Hamiltonian. Such interactions do not therefore fall under the category of LOCC procedures. Consider a situation where the composite particles have no direct interaction between its component particles. The only resource remaining to facilitate the condensation process is then entanglement. If condensation remains possible under such a constraint, then it must be that *entanglement is sufficient for condensation to occur* since direct interaction via an interacting Hamiltonian is not necessary to induce condensation. This fits neatly with the resource view of entanglement: if entanglement is to be considered a resource, then operations that freely increase the amount of entanglement must be excluded.

We will therefore consider a situation where there is initially a "gas" of composite particles, made of a large number N number of composite particles each initially trapped in independent wells which we label by the index n. A composite particle in the nth well is described by the creation operator $c_n^\dagger = \sum_m \sqrt{\lambda_m} a_{n,m}^\dagger b_{n,m}^\dagger$. We say that the condensation has occurred if all the composite particles in this gas is put into a single well (which we choose to be the well $n = 1$) all of which occupy the same state. Written formally, we are considering a state transformation between the two following states:

$$
\begin{aligned}
|\phi\rangle_{ab} &= \prod_n c_n^\dagger |0\rangle &\quad (2.19)\\
|\phi'\rangle_{ab} &= (c_1^\dagger)^N |0\rangle, &\quad (2.20)
\end{aligned}
$$

where $|\phi\rangle_{ab}$ is the aforementioned gas occupying N separate wells, which is the initial state, and $|\phi'\rangle_{ab}$ is the state of N identical composite particles of the same type occupying a single well, which is the final state. The subscript ab reminds us that the state may be partitioned into two portions, one of which describes the fermions of the type a and the other composed of the fermions of the type b. In order for us to make any conclusions regarding the transformations, it will be useful to have the following reduced density matrices:

$$
\begin{aligned}
\rho_a &= \sum_{m_1,\ldots m_N} \lambda_{m_1} \ldots \lambda_{m_N} a_{1,m_1}^\dagger \ldots a_{N,m_N}^\dagger |0\rangle\langle 0| a_{1,m_1} \ldots a_{N,m_N} &\quad (2.21)\\
\rho_a' &= \frac{N!}{\chi_N} \sum_{m_1<\cdots<m_N} \lambda_{m_1} \ldots \lambda_{m_N} a_{1,m_1}^\dagger \ldots a_{1,m_N}^\dagger |0\rangle\langle 0| a_{1,m_N} \ldots a_{1,m_N}, &\quad (2.22)
\end{aligned}
$$

where ρ_a and ρ_a' are the reduced density matrices of $|\phi\rangle_{ab}$ and $|\phi_{ab}'\rangle$ respectively. They are defined to be $\rho_a = \mathrm{Tr}_b(|\phi\rangle_{ab}\langle\phi|)$ and $\rho_a' = \mathrm{Tr}_b(|\phi'\rangle_{ab}\langle\phi'|)$. We also recall that $\chi_N \equiv N! \sum_{m_1<\cdots<m_N} \lambda_{m_1} \ldots \lambda_{m_N}$ (See (1.1.3)).

The ultimate goal is to address the question of whether the transformation from the state

$|\phi\rangle_{ab}$ to $|\phi'\rangle_{ab}$ is always possible using LOCC. For a given composite particle, this is the same as asking if $\rho_a \prec \rho'_a$, due to Thm. 2.4. If it is indeed possible, then we will call the process *LOCC condensation*, and the product a *LOCC condensate*. If a LOCC condensate is possible, then it is possible to have mass occupation of a single state with composite particles where the condensation process is entirely due to the quantum correlations present in the composite particles. Furthermore, if LOCC condensation is *always* possible with sufficiently high entanglement, then this is suggestive evidence that BECs occur in nature because of quantum correlations and that the amount of entanglement offers a good descriptor of a particle's ability to form a condensate. However, it turns out that entanglement is not sufficient to guarantee condensation, and indeed in principle, one is able to construct a composite particle containing an arbitrarily high amount of entanglement and yet still be unable to form a LOCC condensate. This construct will be presented in the following section.

2.2.1 A Composite Particle That Does Not LOCC condense

In this section, we will present a composite particle which does not LOCC condense, but nonetheless possess high levels of entanglement. Again, we consider the composite particle whose state is of the form $c_n^\dagger = \sum_m \sqrt{\lambda_m} a_{n,m}^\dagger b_{n,m}^\dagger$ except this time, we will define the factors λ_m to be the following:

$$\lambda_m = \frac{1}{(m+1)^s \zeta(s)}, \tag{2.23}$$

where $s > 1$ is any fixed constant and $\zeta(s) = \sum_{i=0}^{\infty} \frac{1}{(i+1)^s}$ is the Riemann zeta function. No prior knowledge of the Riemann zeta function is necessary for the argument that follows however. We may verify that the above λ_m defines a valid composite particle by checking that it satisfies the normalization condition $\sum_{m=0}^{\infty} \lambda_m = 1$. Indeed:

$$\sum_{m=0}^{\infty} \lambda_m = \left(\sum_{m=0}^{\infty} \frac{1}{(m+1)^s} \right) / \zeta(s) \tag{2.24}$$
$$= \zeta(s)/\zeta(s) \tag{2.25}$$
$$= 1. \tag{2.26}$$

The amount of entanglement for this composite particle is also easily computed as follows:

$$P = \sum_{m=0}^{\infty} \lambda_m^2 \tag{2.27}$$
$$= \sum_{m=0}^{\infty} \frac{1}{(m+1)^{2s}} / (\zeta(s))^2 \tag{2.28}$$
$$= \zeta(2s)/(\zeta(s))^2. \tag{2.29}$$

Since s is a free parameter, we can choose it such that $s = 1 + \epsilon$ where ϵ is strictly positive but small. We note that for small ϵ, $\zeta(s) \approx 1 + \frac{1}{2} + \frac{1}{3} \ldots$, which is otherwise called the harmonic series. This series is known to be divergent, so in the limit of $\epsilon \to 0$, $\zeta(s)$ approaches being infinitely large. On the other hand, $\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} \cdots = \pi^2/6$ is a known convergent series. This implies that the purity $P = \zeta(2s)/(\zeta(s))^2 \to 0$ in the limit $\epsilon \to 0$. As such, the amount of entanglement in the composite particle, $1 - P$ can be made arbitrarily close to unity. we have therefore demonstrated a class of composite particles which can be made to contain arbitrarily high amounts of entanglement.

We now have a look at whether such a composite particle is able to LOCC condense. To do this, we need to arrange the eigenvalues of ρ_a and ρ'_a. Note first that λ_m is a decreasing function of m for a given s. As such, for ρ_a the largest eigenvalue is given by (See Eqn. (2.21))

$$\lambda_0^N = \frac{1}{\zeta(s)^N}. \tag{2.30}$$

For ρ'_a, the largest eigenvalue is slightly more complicated as no two indices of the indices

m_i can be the same, and it is given by (See Eqn. (2.22))

$$\frac{N!}{\chi_N}\lambda_0 \ldots \lambda_{N-1} = \frac{N!}{\chi_N}\frac{1}{(N!)^s}\frac{1}{\zeta(s)^N}. \tag{2.31}$$

In order for $\rho_a \prec \rho_a'$ we must have that as a first condition(See Eqn. (2.1):

$$\lambda_0^N \leq \frac{N!}{\chi_N}\lambda_0 \ldots \lambda_{N-1}. \tag{2.32}$$

Upon simplifying, it gives us the following:

$$N!^{\epsilon}(\chi_N) \leq 1. \tag{2.33}$$

We then perform a series expansion on the above expression. Keeping in mind that for small ϵ, we have to the first order of ϵ $(N!)^e \approx 1 + (\log N!)\epsilon$ and $\zeta(1+\epsilon) \approx \frac{1}{\epsilon}$, we then have:

$$(N!)^{\epsilon}\chi_N \geq N!(1 - NP) \approx 1 + (\log N!)\epsilon, \tag{2.34}$$

where we used the fact that $\chi_N \geq 1 - NP$ (Eqn. (1.57)). This suggests that for sufficiently small values of ϵ, we are able to get $(N!)^{\epsilon}\chi_N > 1$ thus violating the condition Eqn. (2.33). This allows us to prove via a counter example that even in the case of arbitrarily large entanglement, the possibility of LOCC condensation cannot be guaranteed.

2.2.2 A Composite Particle That Will Always LOCC condense

To complete the discussion of entangled correlations and the condensation process, we will now present a continuous class of composite particles that will *always* LOCC condense for arbitrarily small (but not zero) amounts of entanglement. As before, we will define the coefficients λ_m first:

$$\lambda_m = (1 - z)z^m, \tag{2.35}$$

where $0 < z < 1$ is any fixed constant. The purity of the resulting reduced density matrix of a single composite particle is then given by the following:

$$P = \sum_{m=0}^{\infty} (1-z)^2 z^{2m} = \frac{1-z}{1+z}. \tag{2.36}$$

From the above, we see that the purity can be made arbitrarily close to unity (and entanglement arbitrarily close to zero) by choosing z to be as small as possible. We can also calculate the factor χ_N. In order to do this, we first recall that that $m_1 < \cdots < m_N$. We then define $d_i \equiv m_i - m_{i-1}$ (when $i = 1$, we have the special case $d_1 \equiv m_1$ instead). d_i is the difference between the index m_i and the index before it m_{i-1}. It then becomes apparent that d_i is any positive integer $1, 2, \ldots$ (with the exception of d_1, which can also be zero) since $m_i > m_{i-1}$ and that $m_i = \sum_{j=1}^{i} d_j$. This suggests that we can write χ_N in the following way:

$$\begin{aligned}
\chi_N &= N! \sum_{m_1 < \cdots < m_N} \lambda_{m_1} \ldots \lambda_{m_n} && (2.37)\\
&= N!(1-z)^N \sum_{m_1 < \cdots < m_N} z^{\sum_{i=1}^{N} m_i} && (2.38)\\
&= N!(1-z)^N \sum_{d_2,\ldots,d_N=1}^{\infty} \sum_{d_1=0}^{\infty} z^{\sum_{i=1}^{N} \sum_{j=1}^{i} d_j} && (2.39)\\
&= N!(1-z)^N \sum_{d_2,\ldots,d_N=1}^{\infty} \sum_{d_1=0}^{\infty} z^{Nd_1+(N-1)d_2+\cdots+d_N} && (2.40)\\
&= N!(1-z)^N \left(\sum_{d_1=0}^{\infty} z^{Nd_1} \right) \left(\sum_{(N-1)d_2=1}^{\infty} z^{d_2} \right) \ldots \left(\sum_{d_N=1}^{\infty} z^{d_N} \right) && (2.41)\\
&= N!(1-z)^N \left(\frac{1}{1-z^N} \right) \left(\frac{z^{N-1}}{1-z^{N-1}} \right) \ldots \left(\frac{z}{1-z} \right) && (2.42)\\
&= N!(1-z)^N \frac{z^{N(N-1)/2}}{\prod_{i=1}^{N} (1-z^i)}. && (2.43)
\end{aligned}$$

In order to check that the majorization condition is satisfied, we need to compute and order the eigenvalues of ρ_a and ρ'_a. For ρ_a, if we write the ordered eigenvalues in a vector, it looks like the following:

$$\vec{\Gamma}(\rho_a) \equiv (1-z)^N (1, z, \ldots, z, z^2, \ldots, z^2, z^3, \ldots). \tag{2.44}$$

Notice that each of the terms z^i above may in general be degenerate in the sense that they appear multiple times in the vector. The degeneracy, or the number of times each z^i appears, can be phrased a a combinatorics problem involving the number of ways to divide i balls into N different boxes. Denoting this degeneracy as $g(i)$, it can be verified that $g(i) = \begin{pmatrix} i+N-1 \\ i \end{pmatrix}$. For ρ_a', the vector of eigenvalues also look somewhat similar:

$$\vec{\Gamma}(\rho_a') \equiv \frac{N!(1-z)^N z^{N(N-1)/2}}{\chi_N}(1, z, z^2, \ldots, z^2, z^3, \ldots), \tag{2.45}$$

where again in general, the z^i inside the vector may be degenerate. The exact form of the degeneracy is difficult to phrase precisely, but fortunately knowledge of this is not necessary for our purpose. If we denote $g'(i)$ as the degeneracy of the eigenvalues in ρ_a', we find that $g'(i) \leq g(i)$. This is not difficult to see, and is a result of the fact that the indices m_i summed over in ρ_a' (See Eqn. (2.22)) form a subset of those in ρ_a (See Eqn. (2.21)).

We are now in a position to prove that $\rho_a \prec \rho_a'$. But first, we introduce a few more definitions for notational simplicity:

$$\vec{\gamma}(\rho_a) \equiv \tfrac{1}{(1-z)^N}\vec{\Gamma}(\rho_a) \tag{2.46}$$

$$\vec{\gamma}(\rho_a') \equiv \tfrac{1}{(1-z)^N}\vec{\Gamma}(\rho_a'). \tag{2.47}$$

Notice that the new vectors are simply Eqns. (2.44) and (2.45) sans the common factor $(1-z)^N$. Due to this multiplicative factor, the sum of all the vector elements is now $\sum_{i=1}^{\infty}[\vec{\gamma}(\rho_a)]_i = \sum_{i=1}^{\infty}[\vec{\gamma}(\rho_a')]_i = \frac{1}{(1-z)^N}$, where previously we have instead $\sum_{i=1}^{\infty}[\vec{\Gamma}(\rho_a)]_i = \sum_{i=1}^{\infty}[\vec{\Gamma}(\rho_a')]_i = 1$. The sum of the first k elements of $\vec{\gamma}(\rho_a)$ is given by

$$\sum_{i=1}^{k}[\vec{\gamma}(\rho_a)]_i = \sum_{i=0}^{l-1} g(i) z^i + (k - \sum_{i=0}^{l-1} g(i)) z^l, \tag{2.48}$$

where the l is defined according to the kth element of the vector, which is written in the form $[\vec{\gamma}(\rho_a)]_k = z^l$ for some integer l. We do a similar thing for $\vec{\gamma}(\rho_a')$ and the sum of the first k elements may be written as

$$\sum_{i=1}^{k}[\vec{\gamma}(\rho_a')]_i = f(z)\left(\sum_{i=0}^{m-1} g'(i)z^i + (k - \sum_{i=0}^{m-1} g'(i))z^m\right), \tag{2.49}$$

where $f(z) \equiv \frac{N!z^{N(N-1)/2}}{\chi_N}$ and m is defined by the kth element of $\vec{\gamma}(\rho_a')$, $[\vec{\gamma}(\rho_a)]_k = f(z)z^m$ for some integer m. Since we know that the degeneracy $g'(i) \leq g(i)$, we have that for the same k, $m \geq l$.

However, note that both sums must be equal to each other in the limit $k \to \infty$, so

$$\sum_{i=0}^{\infty} g(i)z^i = \sum_{i=0}^{\infty} g'(i)f(z)z^i. \tag{2.50}$$

We can expand $f(z) = 1+a_1z+a_2z^2\ldots$ in terms of a power series, and note the following:

$$f(z)z^i = z^i + a_1z^{i+1} + a_2z^{i+2}\ldots. \tag{2.51}$$

Observe that the first term in the power series expansion is z^i, so the term $f(z)z^i$ on the right had side of Eqn. (2.49) contributes nothing to lower powers of z (powers less than i). However, in Eqn. (2.50), we have a strict equality so the coefficients in the power series expansion on both sides have to match. This suggests the following:

$$\sum_{i=0}^{m} f(z)g'(i)z^i = \sum_{i=0}^{m} g(i)z^i + O(z^{m+1}), \tag{2.52}$$

where $O(z^{m+1})$ is some positive error term that consists of the higher powers of z. This has to be true in order for the coefficients to match in Eqn. (2.50). If we choose to, adding $g'(i)f(z)z^{m+1}$ to the left hand side of the above equation changes nothing, so we still have $\sum_{i=0}^{m+1} f(z)g'(i)z^i = \sum_{i=0}^{m} g(i)z^i + O'(z^{m+1})$ for some different error term $O'(z^{m+1})$, since $f(z)z^{m+1}$ only contributes to powers $m+1$ or higher. Eqn. (2.52) then leads to the following inequality:

$$\sum_{i=0}^{m} f(z)g'(i)z^i \leq \sum_{i=0}^{m} g(i)z^i, \tag{2.53}$$

which can be used to compare the sum of the first k terms of both vectors together with the fact that $m \geq l$. Consider first the case where $m > l$:

$$\begin{aligned}
\textstyle\sum_{i=1}^{k}[\vec{\gamma}(\rho_a)]_i &= \sum_{i=0}^{l-1} g(i)z^i + (k - \sum_{i=0}^{l-1} g(i))z^l && (2.54)\\
&\leq \sum_{i=0}^{l} g(i)z^i && (2.55)\\
&\leq \sum_{i=0}^{m-1} f(z)g'(i)z^i && (2.56)\\
&\leq f(z)\left(\sum_{i=0}^{m-1} g'(i)z^i + (k - \sum_{i=0}^{m-1} g'(i))z^m\right) && (2.57)\\
&= \sum_{i=1}^{k}[\vec{\gamma}(\rho'_a)]_i && (2.58)
\end{aligned}$$

For the special case where $m = l$, we can verify the following:

$$\begin{aligned}
\textstyle\sum_{i=1}^{k}[\vec{\gamma}(\rho'_a)]_i &= f(z)\left(\sum_{i=0}^{m-1} g'(i)z^i + (k - \sum_{i=0}^{m-1} g'(i))z^m\right) && (2.59)\\
&\geq \sum_{i=0}^{m-1} g(i)z^i + (k - \sum_{i=0}^{m-1} g'(i))z^m && (2.60)\\
&\geq \sum_{i=0}^{m-1} g(i)z^i + (k - \sum_{i=0}^{m-1} g(i))z^m && (2.61)\\
&= \sum_{i=1}^{k}[\vec{\gamma}(\rho_a)]_i. && (2.62)
\end{aligned}$$

The above is sufficient to prove that $\vec{\gamma}(\rho_a) \prec \vec{\gamma}(\rho'_a)$ and that $\rho_a \prec \rho'_a$ for every value of N, and any value of z. This shows that even in the case where there is *vanishing* amounts of entanglement, LOCC condensation is still possible. Therefore, even amongst composite particles that are able to LOCC condense, there is no minimum amount of entanglement necessary in order for condensation to occur. Furthermore, this result reinforces the fact that the amount of entanglement appears to have no bearing on the possibility of forming a condensate.

2.3 Chapter Summary

In this chapter, we have discussed entanglement in the context of boson condensation. By boson condensation, we specifically refer to the ability of bosons to massively occupy a single state. In addressing this issue, we introduce the concept of LOCC condensation, which seeks to transform an initial state where N identical composite bosons occupy N separate potential wells, into a state where all N composite particles are in the same well. We call this final state an LOCC condensate.

The main interest in LOCC condensation is to address the role that quantum correlations (entanglement) play in the role of this typically boson-like phenomena. In considering only LOCC, we divorce the condensation process from inter-particle interactions, leaving only inter-particle correlations as a resource to assist in the condensation process. This is a perspective that is very deeply rooted in quantum informational protocols, where entanglement is typically exploited in the two party setting, where each party performs LOCC-type operations. Our conclusions, proven based on the specifically constructed composite bosons, suggest that entanglement, while necessary to induce a boson-like composite particle and hence condensation, is not sufficient to predict the ability to form a LOCC condensate even when the amount of entanglement is arbitrarily large. This strongly suggests that interaction plays an important role in the condensation process.

On the flipside, we also construct a composite particle which allows for LOCC condensation when the total amount of entanglement is non-zero, but arbitrarily low. Large amounts of entanglement are therefore not necessary to form a condensate. Since entanglement is known to strongly influence how boson-like the algebraic description a composite particle is, we can conclude that while condensation is typically associated with boson activity, one does not need a "strong" boson to induce condensation. "Weak" bosons may in many ways exhibit fermion-like behaviour, but may nevertheless still be able to exhibit massive occupation of a single state. In such a sense, entanglement may be viewed as some sort of a catalyst to the condensation process. It aids the condensation and its presence is necessary and beneficial, but having it does not guarantee the process will occur.

Chapter 3

Measuring Bosonic Behaviour Through Addition and Subtraction

In the previous chapter, we discussed the role that entanglement plays in condensation. Key to the condensation process is that Pauli's exclusion principle does not apply. In order for a gas to form a condensate, a large number N number of particles must be allowed to occupy the same state which in essence is the polar opposite of Pauli's exclusion principle. Entanglement, though important for fermion pairs to exhibit boson-like behaviour, unfortunately does not appear to fully describe condensation, in the sense that it is *possible* even for composite bosons that contain little entanglement to form a condensate. However, this possibility does not necessarily suggest that the process will be *easy*. In this chapter, we will try to make precise what this means by proposing a way to measure Pauli's exclusion.

3.1 Addition and Subtraction Channels

As mentioned before, we are interested in the "difficulty" of producing a given state. In particular, we would like a method of measuring how difficult it is to produce states with two or more composite particles occupying the same eigenstate of the system. This is the regime where Pauli's exclusion principle takes place.

As such, it is necessary for us to take the operational point of view. Operators that we are going to use to describe/transform the state of the systems must first satisfy conditions that make them operationally doable in the laboratory. Fortunately, there is a concise way to characterize what quantum operations are actually allowed within the context of quantum mechanics [10, 22]. We have the following result by Kraus:

Theorem 3.1. *(Kraus' Theorem). If ρ and $\rho' = \mathcal{S}(\rho)$ are density matrices and $\mathcal{S}$ is some quantum operation on ρ which results in ρ', then there exist matrices K_i such that $\rho' = \mathcal{S}(\rho) = \sum_i K_i \rho K_i^\dagger$ which satisfies $\sum_i K_i^\dagger K_i = \mathbb{1}$.*

The above can be proven axiomatically in a precise way. However, if you accept that every quantum mechanical procedure is facilitated by some unitary operation (with a corresponding Hamiltonian) acting on a possibly extended quantum state, we can demonstrate Kraus' Theorem rather more easily:

Proof. Consider first the state ρ. We can always attach a pure state, ancillary system in the state $|0\rangle_E$ which we call the "environment", such that the resulting state is $\rho \otimes |0\rangle_E\langle 0|$. A complete set of states $|m\rangle_E$, $m = 0, 1, \ldots$ describe the environment system. We now perform a joint unitary on this extended system, which results in the final state $U\rho \otimes |0\rangle_E\langle 0|$. We can observe the effect of this unitary interaction on the system of interest ρ by tracing out the environment:

$$\begin{aligned} \mathrm{Tr}(U\rho \otimes |0\rangle_E\langle 0|U^\dagger) &= \sum_m {}_E\langle m|U\rho \otimes |0\rangle_E\langle 0|U^\dagger|m\rangle_E && (3.1) \\ &= \sum_m {}_E\langle m|U|0\rangle_E \, \rho \, {}_E\langle 0|U^\dagger|m\rangle_E. && (3.2) \end{aligned}$$

We now define $K_m \equiv {}_E\langle m|U|0\rangle_E$. Note that ${}_E\langle m|U|0\rangle_E \, \rho \, {}_E\langle 0|U^\dagger|m\rangle_E$ are positive, since it is just the (unnormalized) state after performing a measurement with outcome m on the environment. As a result, we see that $K_m^\dagger K_m$ is also, since ρ can be any state in general, so we have $\mathrm{Tr}(K_m|\psi\rangle\langle\psi|K_m^\dagger) = \langle\psi|K_m^\dagger K_m|\psi\rangle \geq 0$.

We can verify the following:

$$\begin{aligned}
\textstyle\sum_n K_m^\dagger K_m &= \sum_m {}_E\langle 0|U^\dagger|m\rangle\langle m|U|0\rangle_E && (3.3)\\
&= {}_E\langle 0|U^\dagger U|0\rangle_E && (3.4)\\
&= \mathbb{1}. && (3.5)
\end{aligned}$$

So indeed, K_m forms a set of Kraus operators. □

For a more complete discussion of Kraus' Theorem, see [10]. We now consider an ideal creation and annihilation operator of a boson $a^\dagger$ and a. These operators represent the operation of adding and subtracting a boson from the system, but are they physically implementable? Unfortunately, it can be demonstrated that $a^\dagger$ and a are not physically realizable quantum operations. Let us suppose that $a^\dagger$ is a valid Kraus operator such that $K_0 = a^\dagger$. Since $\sum_i K_i^\dagger K_i = \mathbb{1}$, we have:

$$\sum_{m\neq 0} K_i^\dagger K_i = \mathbb{1} - K_0^\dagger K_0 = \mathbb{1} - aa^\dagger. \qquad (3.6)$$

The left hand side of the equation is a sum of positive operators, but the right hand side of the equation can be negative. This is because $aa^\dagger$ has eigenvalues $N+1$ where $N = 0, 1, \ldots$, so $\mathbb{1} - aa^\dagger$ must have eigenvalues $-N$. We therefore have to conclude that $a^\dagger$ is not an operationally performable quantum procedures. A similar argument applies for the annihilation operator a. We can also safely conclude that the operators a and $a^\dagger$, even if implementable, must correspond to probabilistic procedures as they can locally increase the amount of entanglement if they are deterministically performed. Consider an initial bipartite state shared between Alice and Bob, $|\psi\rangle_{ab} = \frac{1}{\sqrt{3}}(|01\rangle_{ab} + \sqrt{2}|10\rangle_{ab})$. The entropy of entanglement is clearly less than 1. If Bob adds a particle on his side by performing $a^\dagger$, the resulting state is $|\psi\rangle_{ab} = \frac{1}{\sqrt{2}}(|12\rangle_{ab} + |11\rangle_{ab})$ which is unitarily equivalent to a Bell state, and so has entanglement equal to 1! Since locally increasing the amount of entanglement deterministically is impossible, we therefore have to conclude that $a^\dagger$ cannot be deterministically performed; there has to be outcomes where the total entanglement is decreased so that entanglement does not increase on average. A similar argument also applies for the annihilation operator a.

We therefore seek to construct the Kraus operator equivalent of $a^\dagger$ and a. Consider an operator that effectively adds a particle to a system:

$$a^\dagger_{\text{eff}} = \sum_{n=0}^{N_{max}} f_n |n+1\rangle\langle n|. \tag{3.7}$$

We note that that unlike the ideal creation operator $a^\dagger$ which is able to add a particle to any system with any arbitrarily large number of particles N, the above operator only acts on systems with particle numbers up to a maximum of N_{max} number of particles. This may be physically motivated by the fact that in a physically realizable setting, there is always a finite amount of resources. Therefore, a system that an experimenter is operating on always has an upper bound on the number of particles it contains.

In order for the operator to retain the properties of the the ideal bosonic creation operator, have have to have that $\frac{f_n}{f_{n-1}} = \frac{\sqrt{n+1}}{\sqrt{n}}$, so as to respect the ratios of the coefficients given by $a^\dagger|n\rangle = \sqrt{n+1}|n\rangle$. This necessarily implies that $f_n = g\sqrt{n+1}$ for some fixed constant g. Recall that the issue with implementing $a^\dagger$ is that the smallest eigenvalue of $\nVdash - aa^\dagger$ is negative. It is easy to see that the smallest eigenvalue of $\nVdash - a_{\text{eff}} a^\dagger_{\text{eff}}$ is $1 - g^2(N_{\text{max}} + 1)$, so to ensure positivity, we have to have

$$g \leq \frac{1}{\sqrt{N_{\text{max}+1}}}. \tag{3.8}$$

Any value of g within this range will be able to successfully emulate the effect of an ideal creation operator. The probability of successfully performing this operation is then be verified to be given by

$$(\text{Probability of implementation}) = \text{Tr}(a_{\text{eff}} a^\dagger_{\text{eff}} \rho) = g^2(N+1)\text{Tr}\left(|n\rangle\langle n|\rho\right), \tag{3.9}$$

so the operation has the highest chance of success by choosing $g = \frac{1}{\sqrt{N_{\text{max}+1}}}$. Once the effective creation operator is well defined, we immediately see that its conjugate is the annihilation operator,

$$a_{\text{eff}} = \sum_{n=0}^{N_{\max}} g\sqrt{n+1}|n\rangle\langle n+1| = \sum_{n=1}^{N_{\max}+1} g\sqrt{n}|n-1\rangle\langle n|, \tag{3.10}$$

from which it can be seen that it correctly removes a particle and contains the correct coefficients. We therefore have the correct description of the creation and annihilation operators in terms of operationally achievable Kraus operators.

3.2 Measuring Bosonic and Fermionic Quality

In this section we will present a proposal to measure the bosonic and fermionic quality of a given particle. In the previous chapters, we have studied the the role that the quantity $\chi_2 = 1 - p$ plays in quantifying the bosonic nature of composite bosons, so a natural question to ask may be why there is even a need for an alternative method to measure bosonic quality. The reason for this is that χ_2 requires some implicit assumption regarding the internal structure of the particle. The usefulness of χ_2 as a method of quantifying bosonic and fermionic quality depends on the assumption that the internal structure of the particle is composed of 2 correlated fermions. There are many composite bosons that do not fall under this category, and as such, it is preferable that a new method of quantification is introduced that can be more generally applied.

Also, when dealing with microscopic objects, the internal structure of a given particle can be ambiguous. Any time any given experiment fails to reveal the internal structure of a particle, it is always uncertain whether this is a limitation of the experiment or if the particle is truly elementary. Our proposed quantity attempts to sidestep these difficulties by basing them entirely on elementary operations, which in principle can be performed regardless of the type of particle being considered.

Our proposed measure is based on particle addition and subtraction, since these operations are elementary enough to apply to all particle types. Consider a stochastic mixture of different numbers of particles all occupying the same mode, $\rho = \sum_n p_n|n\rangle\langle n|$. We will first add a particle to the system, then subtract a particle from them by applying the (effective) creation and annihilation operators respectively. The proposed method is flexible enough to be adaptable to any such state ρ that an experimentalist may prepare, but keeping in mind that specifically for ideal fermions, it is not possible to prepare more than 1 particle occupying a single mode, we will consider only states of the form $\rho = p_0|0\rangle\langle 0| + (1-p_0)|1\rangle\langle 1|$ so that the defined measure will be completely general.

We begin by defining the following quantity:

$$M = p_0 - p_0^{\mathrm{AS}}, \tag{3.11}$$

where p_0 is the initial probability of finding the system in the vacuum state, and p_0^{AS} is the probability of a vacuum state post addition and subtraction operations. Correspondingly, we will denote the state post addition and subtraction ρ^{AS}. Suppose instead of adding and subtracting a quantum indistinguishable particle, we perform what is essentially a classical procedure and add and subtract a classical, distinguishable particle. Classical addition and subtraction does not influence the probability distribution of the final state so $M = 0$. This defines for us the point of classicality and is independent of the distribution of the initial state.

For ideal bosons, it is can be verified that suppose the addition and subtraction operations are successful, then the final normalized state is simply given by $\rho^{\mathrm{AS}} = \frac{1}{4-3p_0}(p_0|0\rangle\langle 0| + 2^2(1-p_0)|1\rangle\langle 1|)$, which gives us the following:

$$M = \frac{3p_0(1-p_0)}{4-3p_0}. \tag{3.12}$$

For an ideal fermion, we have $\rho^{\mathrm{AS}} = |0\rangle\langle 0|$, therefore

$$M = -(1-p_0). \tag{3.13}$$

We see that if we set $p_0 = \frac{2}{3}$, we have for bosons $M = 1/3$ and for fermions $-1/3$. From this, we can define our *standard measure*, based on addition and subtraction on the *standard state* $\rho = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 0|$:

$$\mathcal{M} = 3M \tag{3.14}$$

The points $\mathcal{M} = -1, 0, 1$ correspond to ideal fermions, classical distinguishably, and ideal bosons respectively. You would expect that as a result $-1 \leq \mathcal{M} \leq 1$ and this intuition is indeed correct for composite bosons made of 2 distinguishable fermions, as we shall later see. This is, however, not the maximum range of possible values of $\mathcal{M}$. Let us rewrite p_0^{AS} in the following manner:

$$p_0^{\mathrm{AS}} = \frac{\frac{2}{3}r}{\frac{2}{3}r + \frac{1}{3}r} = \frac{2}{2 + \frac{s}{r}}, \tag{3.15}$$

where $r \equiv \mathrm{aa}^\dagger|0\rangle\langle 0|\mathrm{aa}^\dagger$ and $s \equiv \mathrm{aa}^\dagger|1\rangle\langle 1|\mathrm{aa}^\dagger$. As a result, we can verify the following:

$$\mathcal{M} = 3(p_0 - p_0^{\mathrm{AS}}) = \frac{2(\frac{s}{r} - 1)}{2 + \frac{s}{r}}. \tag{3.16}$$

The quantities r and s have a probabilistic interpretation so must remain positive, but if we do not otherwise presuppose any constraints on them, the resulting range of values is $-1 \leq \mathcal{M} < 2$. It may very well be that the entire range of $\mathcal{M}$ is not reachable due to other physical considerations, but this surprisingly turns out not to be so and the entire range of $\mathcal{M}$ can be accessed. This will be discussed in the subsequent sections.

3.3 The Standard Measure and Composite Bosons

In order to confirm that the above proposed measure makes sense, we check consistency using what we already know about composite bosons. From the previous chapters, we have already established that the "boson-ness" of a composite particle of 2 non-identical fermions may be directly related to the quantity $\chi_2 = 1 - P$ (See Section 1.2.1). In order to establish consistency, we need to verify that the measure $\mathcal{M}$ is an increasing function of χ_2, or alternatively, a decreasing function of the purity P.

Consider first the creation operator describing a composite boson of this type:

$$c^\dagger = \sum_m \sqrt{\lambda_m} a_m^\dagger b_m^\dagger, \tag{3.17}$$

where $a^\dagger$ and $b^\dagger$ correspond to creation operators or two different types of fermions. For similar reasons as ideal creation operators, this is not a realistic quantum operation since it is not strictly a Kraus operator. Nevertheless, just as for ideal boson creation operators, we can construct a valid Kraus operator which effectively performs it. We first construct the relevant n composite particle state through the following:

$$|n\rangle = \frac{1}{\sqrt{\chi_n}\sqrt{n!}}(c^\dagger)^n|0\rangle. \tag{3.18}$$

From this, we define the effective creation operator to be the following, in a process similar to Eqn. (3.7):

$$c^\dagger_{\text{eff}} = \sum_{n=0}^{N_{\max}} g\sqrt{\frac{\chi_{n+1}}{\chi_n}}\sqrt{n+1}|n+1\rangle\langle n|. \tag{3.19}$$

It can be verified that the effective operator essentially functions identically to the ideal creation operator $c^\dagger$ over the range of particle numbers up to $N_{\max}$ and up to a common factor g. The factor g essentially characterizes how probable one is able to successfully add a particle, compared to how probable it is to *fail* to add a particle. If one were to consider only the subset of data where the particle addition was successful, it can be verified that the effect of g on the probabilities will always be cancelled out so it doesn't matter what value it is, although experimentally it is favorable to make it as large as possible. Henceforth, we will always assume that this is done.

From the above creation operator, the corresponding annihilation operator is simply given by its hermitian conjugate c_{eff}. It can be verified that its effect on $|n\rangle$ is

$$c_{\text{eff}}|n\rangle = \sqrt{\frac{\chi_n}{\chi_{n-1}}}\sqrt{n}|n-1\rangle. \tag{3.20}$$

One may observe that this differs from the ideal operator c since $c|n\rangle = \sqrt{\frac{\chi_n}{\chi_{n-1}}}\sqrt{n}|n-1\rangle + |\epsilon_n\rangle$ (See Eqn. (1.32)) due to the missing "trash term" $|\epsilon_n\rangle$. The reason for this is because, as previously discussed for particle addition, we only consider the subset of data/experiments where the particle subtraction is successful. This involves checking the number of composite particles in the system after performing the operation. Since $|\epsilon_n\rangle$ is orthogonal to all the n particle states $|n\rangle$, its occurrence is considered a failed particle subtraction.

Using the fact that $c_{\text{eff}}^{\dagger}|1\rangle = \sqrt{2\chi_2}|2\rangle$ and $c_{\text{eff}}|2\rangle = \sqrt{2\chi_2}|1\rangle$, we get

$$p_0^{\text{AS}} = \frac{\frac{2}{3}}{\frac{2}{3} + \frac{1}{3}(2\chi_2)^2} \tag{3.21}$$

$$= \frac{1}{1 + 2\chi_2^2}. \tag{3.22}$$

As a result, we have for the standard measure

$$\mathcal{M} = 2 - \frac{3}{1 + 2\chi_2^2}. \tag{3.23}$$

The above expression for $\mathcal{M}$ is a monotonically increasing function of χ_2. Since $0 \leq \chi_2 \leq 1$, we have that for composite particles of 2 non-identical fermions, $-1 \leq \mathcal{M} \leq 1$, where $\mathcal{M} = -1$ is achieved in the limit $\chi_2 \to 0$ and $\mathcal{M} = 1$ is achieved in the limit $\chi_2 \to 1$. This effectively reproduces what was previously already known regarding composite bosons of this type. Namely that in the limit of no entanglement the composite particle behaves more like a fermion, and in the limit of infinite entanglement the particle behaves more like a boson. Note also that our proposed measure directly tells us the purity of the composite boson. Measuring the quantity $\mathcal{M}$ therefore simultaneously provides information regarding the purity and the amount of entanglement within the composite boson.

3.4 Systems of 2 Distinguishable Bosons

In the previous section, we demonstrated that composite bosons may achieve $-1 \leq \mathcal{M} \leq 1$ thus bridging the gap between ideal fermions and ideal bosons. In this section, we will consider a system that will achieve values of $\mathcal{M}$ *beyond* ideal bosons. In order to achieve this, we will consider a system of 2 correlated, non-identical bosons (not to be confused with composite bosons, which in this thesis are actually composite particles made up of fermions but exhibit boson-like properties). The creation operator is similarly given by

$$c^\dagger = \sum_m \sqrt{\lambda_m} a_m^\dagger b_m^\dagger, \tag{3.24}$$

where the operators $a^\dagger$ and $b^\dagger$ now correspond to 2 different types of bosons instead. Again, we will construct the effective creation operator

$$c_{\text{eff}}^\dagger = \sum_{n=0}^{N_{\max}} g \sqrt{\frac{\chi_n + 1}{\chi_n}} \sqrt{n+1} |n+1\rangle\langle n|, \tag{3.25}$$

from which we will achieve almost identical relations $c_{\text{eff}}^\dagger|1\rangle = \sqrt{2\chi_2}|2\rangle$ and $c_{\text{eff}}|2\rangle = \sqrt{2\chi_2}|1\rangle$, which the exception that χ_2 is now instead:

$$\chi_2 = 2 \sum_{m \geq n} \lambda_m \lambda_n = \sum_{m,n} \lambda_m \lambda_n + \sum_m \lambda_m^2 = 1 + P. \tag{3.26}$$

Note that this differs from the case of composite bosons where $\chi_2 = 1 - P$. Nevertheless, the expression for $\mathcal{M}$ remains identical:

$$\mathcal{M} = 2 - \frac{3}{1 + 2\chi_2^2}. \tag{3.27}$$

Except that now, $1 \leq \chi 2 \leq 2$, so the maximum achievable value of $\mathcal{M}$ for boson pairs is $\frac{5}{3}$.

One may choose to go even further. For instance, it is possible to construct correlated systems using even larger number of bosons. We can consider correlated systems of k different bosons with creation operators of the type

$$c^\dagger = \sum_m \lambda_m a_m^{(1)\dagger} \dots a_m^{(k)\dagger}, \tag{3.28}$$

where $a^{(i)\dagger}$ for $i = 1, \dots, k$ stands for different boson creation operators if the index i is different. Unless $k = 2$ and any composite particle can always be written in the above form due to the Schmidt decomposition, the above state is a special case. Regardless, it can be verified that the measure $\mathcal{M}$ is still given by Eqn. (3.27), so we will only need to compute χ_2. This is especially simple for a composite particle of this type, and is given by

$$\begin{aligned}
\chi_2 &= \frac{1}{2}\langle 0|c^2(c^\dagger)^2|0\rangle && (3.29)\\
&= 2\sum_{m>n}\lambda_m\lambda_n + 2^{k-1}\sum_m \lambda_m^2 && (3.30)\\
&= \sum_{m\neq n}\lambda_m\lambda_n + 2^{k-1}\sum_m \lambda_m^2 && (3.31)\\
&= \sum_{m,n}\lambda_m\lambda_n + (2^{k-1}-1)\sum_m \lambda_m^2 && (3.32)\\
&= 1 + (2^{k-1}-1)P, && (3.33)
\end{aligned}$$

where P is the purity of the reduced density matrix of any one of the bosons that make up the composite particle. From this is is clear that as the number of different bosons increase $k \to \infty$, then $\chi_2 \to \infty$ and $\mathcal{M} \to 2$. We can therefore conclude that in principle, the entire range of values $-1 \leq \mathcal{M} < 2$ is accessible by some composite particle.

3.5 Interpreting Entanglement

We now return to the case where the composite particle is composed of either 2 bosons or two fermions. A comparison of these two scenarios allows us to form an intuitive explanation what what entanglement really does.

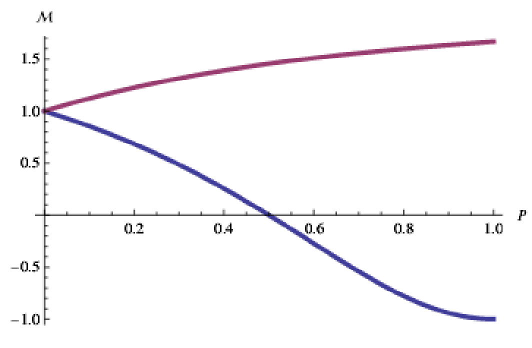

Figure 3.1: The plot of $\mathcal{M}$ against the purity P. Top curve is for composite particles of 2 bosons, bottom curve is for composite particle of 2 fermions.

From Fig. (3.1) above, some conclusions can be made. Note that the higher the purity P, the less entangled the system. For composite particles made up of 2 fermions, we see that the particle approaches being an ideal fermion when entanglement is low, and approaches being an ideal boson when entanglement is high. This is already expected, based on the discussion in the previous chapters.

For composite particles made up of 2 bosons however, we see that instead of increasing the value of $\mathcal{M}$ with increasing entanglement, we have the opposite: $\mathcal{M}$ decreases as the amount of entanglement increases. The endpoint is the same as in the 2 fermion case however. In the limit of infinite entanglement, a composite particle of 2 bosons will approach being a *single* ideal boson, as opposed to being a system of 2 bosons. The fact that 2 or more bosons, even if uncorrelated, can achieve $\mathcal{M} > 1$ is also now relatively intuitive. The composite particle is "more bosonic" than a single ideal boson because it itself contains more than one boson.

In this comparison, it is now clear what role entanglement plays in a composite particle. Entanglement is not a a resource that creates "boson-ness". Instead, it is a resource that enables multiple particles to behave as a single particle. 2 fermions with half integer spins, when sufficiently entangled, will behave like a single boson with integer spin. 2 bosons with integer spins, when sufficiently entangled, will stop behaving like multiple bosons, and start behaving like a single boson with integer spin. As such, entangled correlations is more accurately described as a resource which erases the "individuality" of the component particles, and creates a single particle out of the composite.

This assertion is entirely consistent with what is known regarding entanglement in quantum information theory. Entangled states are a necessary resource in order to achieve *nonlocality*, which asserts that there does not exist local hidden variables that describes quantum mechanical subsystems [23–25]. Since no hidden variables can describe each subsystem completely, we are forced to conclude that in an entangled system, its components possess no individual reality. It is precisely this property that comes into play in composite particles.

3.6 Two Particle Interference

In this section, we see that the measure $\mathcal{M}$ is in some ways equivalent to a bunching and anti bunching experiment performed using beam splitters.

Consider ideal boson creation operators $a^\dagger(L)$ and $a^\dagger(R)$. L, R correspond to the "left" and the "right" path that the boson may enter the beam splitter from. Then a beam splitter Hamiltonian looks like the following:

$$H_{\text{BS}} = a^\dagger(L)a(R) + a^\dagger(R)a(L). \tag{3.34}$$

If the beam splitter hamiltonian H_{BS} is applied for a duration $t = \frac{\pi}{4}$, we the final state is given by

$$\exp(-i\ H_{\text{BS}}\ t)|1_L, 1_R\rangle = i\left(\frac{|2_L, 0_R\rangle + |0_L, 2_R\rangle}{\sqrt{2}}\right), \tag{3.35}$$

where the interpretation is that two bosons entering the beam splitter simultaneously from different sides will always exit the beam splitter on the same side, either left or right. This phenomena is called *bunching* [26]. Bunching can clearly only be supported by bosons, since ideal fermions obey Pauli's exclusion principle and which will prevent them appearing on the same side of the beam splitter. As such, bunching is usually considered a bosonic phenomena.

For general time t, the above Hamiltonian will produce the following state:

$$\exp(-i\ H_{\text{BS}}\ t)|1_L, 1_R\rangle = \cos(2t)|1_L, 1_R\rangle + i\sin(2t)\left(\frac{|2_L, 0_R\rangle + |0_L, 2_R\rangle}{\sqrt{2}}\right), \tag{3.36}$$

where $\sin^2(2t)$ is the probability that bunching will occur. Let us now consider the case for a composite particle of 2 distinguishable fermions. In analogy with the beam splitter Hamiltonian for ideal bosons, we construct the following beam splitter-like Hamiltonian for composite particles from the effective operators defined in Eqn. (3.19). The Hamiltonian is given by

$$H_{\mathrm{BS}} = c_{\mathrm{eff}}^{\dagger}(L) c_{\mathrm{eff}}(R) + c_{\mathrm{eff}}^{\dagger}(R) c_{\mathrm{eff}}(L). \tag{3.37}$$

The effect of such a Hamiltonian for the duration t produces the following state:

$$\exp(-i\ H_{\mathrm{BS}}\ t)|1_L, 1_R\rangle = \cos(2\sqrt{\chi_2}t)|1_L, 1_R\rangle + i \sin(2\sqrt{\chi_2}t) \left(\frac{|2_L, 0_R\rangle + |0_L, 2_R\rangle}{\sqrt{2}} \right). \tag{3.38}$$

Observe that the state allows for bunching to occur so long as the amount of entanglement (i.e. χ_2) is non-zero. Taken from this perspective, every pair of correlated, distinguishable fermions may be considered bosons, because they allow for bunching. However, the amount of time required for bunching to occur deterministically with probability 1 under such a Hamiltonian is $t_{\mathrm{bunch}} = \frac{\pi}{4\sqrt{\chi_2}}$. In the limit of $\chi_2 \to 0$, the amount of time required for bunching to occur goes to infinity. Therefore, the amount of entanglement is not especially relevant when considering whether bunching is *possible*, but it is important in characterizing how difficult it is to produce bunching, in the sense that a longer interaction is required in order to produce the required result. Recall that the measure $\mathcal{M}$ is a monotonically increasing increasing function of χ_2, and so is $1/t_{\mathrm{bunch}}$. They are therefore equivalent measures and hold the same information content.

To complete the discussion on beam splitters, we perform the same analyses for composite particles of 2 distinguishable bosons. It can be verified that the resulting state is again given by Eqn (3.38), with the exception that $\chi_2 = 1 + P$ and is no longer a measure of entanglement (we measure entanglement using $1 - P$). Regardless, $t_{\mathrm{bunch}} = \frac{\pi}{4\sqrt{1+P}}$, from which we see that $\mathcal{M}$ and $1/t_{\mathrm{bunch}}$ are both monotonically decreasing as entanglement increases and $P \to 0$. As such, both $\mathcal{M}$ and $1/t_{\mathrm{bunch}}$ contain the same information content.

3.7 Chapter Summary

In this chapter, we proposed and analyzed a measure to quantify the bosonic/fermionic behaviour in particles. The measure is based on particle addition and subtraction operations. In order to do this, we defined Kraus operators that optimally implements addition and subtraction channels. The primary benefit of basing a measure on addition and subtraction is that these are elementary operations that apply to any particle under consideration. The proposed measure applies to any particle type, and is not based on a property that is peculiar to a single particle species. We also show that every possible value of our measure, denoted $\mathcal{M}$, can be reached by some composite particle, and identify the points on the measure that correspond to ideal fermions, distinguishable particles, and ideal bosons.

In the process of analyzing particle behaviours after implementing the addition and subtraction operations, we also identify the role that entanglement plays in composite particles. Entanglement is not a resource to increase "boson-ness" nor "fermion-ness". Rather, it is a resource to create a single particle out of many. Increasing entanglement leads to behaviour that is increasingly like that of a single particle. As you increase the amount of entanglement between them, a pair of fermions with half-integer spins increasingly behave more like a single boson with integer spin. Similarly, with increasing entanglement, a pair of bosons, instead of becoming even more bosonic, become *less* bosonic instead, since it starts to adopt characteristics of a single boson, rather than multiple bosons.

As a final note, there also exists alternative measures of bosonic behavior. See for example [27–29].

Chapter 4

Entanglement, Composite Particles, and the Szilard Engine

In this chapter, we will discuss the thermodynamics of composite particles in the context of the Szilard engine. First proposed as a refinement of the concept of Maxwell's demon [30, 31], the Szilard engine is an idealized heat engine which uses information to produce work. Superficially, if this process is allowed, it will also seemingly allow for the violation of the second law of thermodynamics, which prohibits an overall decrease in the amount of entropy. This may be circumvented if the measurement process or memory erasure is associated with an increase in entropy [32–35]. It is now widely accepted that the measurement process, including the erasure of memory requires a minimum energy cost of at least $k_BT \ln 2$, where k_B is the Boltzmann constant and T is the thermodynamic temperature.

We are primarily interested in analyzing composite particles within the context of the Szilard engine. Before we begin, however, we will first analyze the fully quantum version of this engine.

4.1 The Quantum Szilard Engine

In this section, we will develop the quantum version of the Szilard engine, first done by Kim *et al.* [36]. This will go on to establish the framework necessary to continue the analysis further using composite particles.

Consider a system described by a Hamiltonian H satisfying $H|n\rangle = E_n|n\rangle$. We define the internal energy U of the system to be

$$U \equiv \sum_n E_n P_n, \tag{4.1}$$

where P_n is the mean occupation number of the nth eigenstate of the system. By computing the derivative, we get the following:

$$dU = \sum_n (E_n dP_n + P_n dE_n). \tag{4.2}$$

In classical thermodynamics, we have the following:

$$dU = Tds - dW, \tag{4.3}$$

where Tds is associated with the heat flow into the system and dW is the work done by the system. In analogy with this, we do the same for Eqn. (4.2). Consider a cylinder filled with a gas with a movable piston on one end. By moving the piston, we are changing the energy spectrum of the system. This is similar to changing the width of a square well potential – different widths will lead to a different spectrum. As such, the change in internal energy due to the change in energy spectrum dE_n must be associated with the work done by the system, $dW = -\sum_n P_n dE_n$. Correspondingly, suppose the piston is not moving and the internal energy changes. This change must have been facilitated by heat exchange between the gas and its surroundings. Therefore, $E_n dP_n$ is to be associated with Tds. Since this is an engine, we are primarily interested in the work done by the system.

In order to calculate the amount of work extracted from the engine, we need the partition function $Z \equiv \mathrm{Tr}e^{-\beta H} = \sum_n e^{-\beta E_n}$ where $\beta \equiv \frac{1}{k_B T}$ is the inverse temperature. From this, we get $P_n = e^{\beta E_n}/Z = -k_B T \frac{\partial \ln Z}{\partial E_n}$, so $dw = \sum_n k_B T \frac{\partial \ln Z}{\partial E_n} dE_n$. Suppose the energy spectrum depends on some external parameter X, so $E_n = E_n(X)$, then the work done as a result of a change in this external parameter from initial state X_1 to X_2 quasistatically is given by

$$
\begin{aligned}
W &= \int_{W(X_1)}^{W(X_2)} dW &(4.4)\\
&= \sum_n k_B T \int_{E_n(X_1)}^{E_n(X_2)} \frac{\partial \ln Z}{\partial E_n} dE_n &(4.5)\\
&= k_B T \sum_n \int_{X_1}^{X_2} \frac{\partial \ln Z}{\partial E_n}\frac{\partial E_n}{\partial X} dX &(4.6)\\
&= k_B T \sum_n \int_{X_1}^{X_2} \frac{dZ}{dX} dX &(4.7)\\
&= k_B T(\ln Z(X_2) - \ln Z(X_1)). &(4.8)
\end{aligned}
$$

The amount of work performed is therefore dependent only on the initial and final state of the external parameter. For example, in the case of the cylinder with movable piston on one end, the work performed will depend only on the initial and final position of the piston, so long as the change in position is performed quasistatically. We will now use Eq. 4.4 to compute the work done by a Szilard Engine.

We begin by first describing a Szilard engine. Consider a closed box containing N particles initially in thermodynamic equilibrium. The engine then works according to the following 5 step process (See (4.1) for illustration):

1. A large box of size L is initially prepared containing N particles and placed in a heat bath at temperature T. The box is in thermal equilibrium.

2. A movable wall is inserted quasistatically into the large box at some position l and then fixes its position. The wall must be a potential barrier sufficiently wide w.r.t. the particle, such that when it is fully inserted you effectively partition the initial large box into 2 boxes. When the wall is fully inserted, we will assume it is impenetrable.

3. A measurement is performed on the system by the experimenter, also sometimes referred to as a "demon", counting the number of particles on the left and on the right side of the wall.

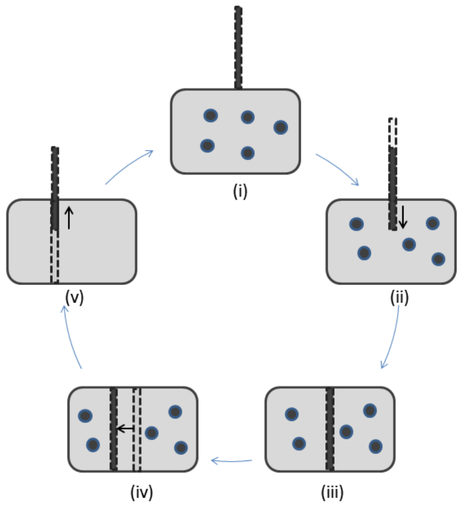

Figure 4.1: An illustration of the cyclic process in the Szilard Engine. (i) The box is initially in thermal equilibrium with its surroundings. (ii) A wall is inserted at some point along the box. (iii) Upon full insertion of the wall, the position of the wall is fixed and a measurement is performed on the system, rotating the entire system if necessary in order to extract work. (iv) The wall is allowed to move along the box, and eventually moves to an equilibrium position, performing work in the process. (v) The wall is removed and the box is allowed to equilibrate, returning it to the state in step (i), where the cycle begins anew.

4. Based on the measurement outcome, the demon will know if the pressure is greater on the left or the right. Based on this information, he will rotate the box such that the side with the greater pressure is always on the left. He then unfixes the wall, allowing it to move left and right so that equilibrium is achieved.

5. The wall is removed quasistatically. Following which the cycle can begin again from stage 1.

Only stages 2, 4 and 5 will involve performing some work. They correspond to the insertion, expansion and removal phases of the engine.

We now consider the wall insertion phase. Before the insertion of the wall, the partition function is given by $Z = \sum_n e^{-\beta E_n}$. The wall is then inserted quasistatically at position l along the box and the system is always maintained in the state of thermal equilibrium, right up until the the wall is fully inserted and has its position fixed. Suppose after the wall insertion is completed, a measurement finds the system with m particles to the left of the partition, and $N - m$ particles to the right of the partition. We denote the corresponding partition function $Z_m(l)$. The partition function before this measurement but after the wall insertion is given by $Z(l) \equiv \sum_{m=0}^{N} Z_m(l)$. As such, the work required for the insertion process is

$$W_{\text{ins}} = k_B T \left[\ln Z(l) - \ln Z\right]. \tag{4.9}$$

After the insertion phase, we perform the measurement, for which the system is not required to perform any work. As mentioned previously, the measurement finding m particles to the left will result in the partition function $Z_m(l)$.The probability of such a measurement is given by $p_m = Z_m(l)/Z(l)$. The wall is now allowed to move along the box and the system will eventually equilibrate at some point along the box when the pressure on both sides of the box is equal (at which point it equilibrates will depend on the box itself, and m). Suppose the final position of the wall, for a given m is l_m. The work required for a given m is $k_B T \left[\ln\left(Z_m(l_m)\right) - \ln\left(Z_m(l)\right)\right]$. Therefore, the average amount of work required for the expansion phase is

$$W_{\text{exp}} = k_B T \sum_{m=0}^{N} p_m \left[\ln\left(Z_m(l_m)\right) - \ln\left(Z_m(l)\right)\right]. \tag{4.10}$$

After the expansion phase, the position of the wall is locked again, and we will begin the wall removal phase, which is the last stage of the cycle. Let us suppose the external parameter describing this change is X, where the initial and final value of this parameter is X_1 and X_2 respectively and $X_1 < X_2$. Then, at some point X_t such that $X_1 < X_t < X_2$, the wall is sufficiently lowered such that the particles are allowed to cross over the partition. This suggests that when the parameter X is less than X_t, the partition function remains $Z_m(l_m)$ since tunneling effects are not significant, and no work is performed during this process. When $X = X_t$ however, tunneling over the wall occurs and the resulting partition function is $Z(l_m) = \sum_{n=0}^{N} Z_n(l_m)$. At the end of the process, the wall is completely removed, so the final partition function is simply Z. The total work done during the wall removal process is given by

$$W_{\text{rem}} = k_B T \sum_{m=0}^{N} p_m \left[\ln Z - \ln\left(Z(l_m)\right)\right]. \tag{4.11}$$

The total work done, given by the sum of work performed during all three processes, is then

$$
\begin{aligned}
W &= W_{\text{ins}} + W_{\text{exp}} + W_{\text{rem}} &(4.12)\\
&= k_B T \sum_{m=0}^{N} p_m \ln \frac{p_m}{p_m^*}, &(4.13)
\end{aligned}
$$

where $p_m^* \equiv \frac{Z_m(l_m)}{Z(l_m)}$.

In principle, the above expression is computable given complete specification of the system. For the case where $N = 2$ and there is a symmetric trapping potential where the wall is inserted from the middle (i.e. a symmetric "box" where the wall is inserted at position $L/2$), the expression is greatly simplified. If two particles are on the left or on the right of the wall, at equilibrium, the wall will be pushed to the ends of the box, so we have $Z_0(l_0) = Z_2(l_2) = Z$. Thus,

$$p_0^* = p_2^* = 1. \tag{4.14}$$

Furthermore, for the case where one particle is on each side of the partition, the equilibrium position is simply in the middle of the box, which is the point where the wall inserted. So $p_1 = p_1^* = Z_1(\frac{L}{2})/Z$. As a result, we have the following concise expression for the work done by the engine:

$$W = -2k_B T p_0 \ln p_0, \tag{4.15}$$

which is an increasing function of p_0, the probability that both particles are on one side of the partition.

4.2 The Probability of Producing an N Particle State

Before we begin an analysis of the Szilard engine in the context of composite particles, we first provide a physical interpretation of χ_N (See Eqn. (1.31)), which will be important in the subsequent sections.

Once again, we will consider composite particles made up of 2 distinguishable fermions, described by the creation operator $c^\dagger = \sum_m \sqrt{\lambda_m} a_m^\dagger b_m^\dagger$. From this, we get a series of normalization factors

$$\chi_N := \|(c^\dagger)^N|0\rangle\|^2/N!. \tag{4.16}$$

One may interpret this factor in terms of relative probabilities. As mentioned in the previous chapter, the operation of adding a particle to the system is inherently probabilistic. The factor χ_N determines the relative probability of success of creating the (unnormalized) state $(c^\dagger)^N|0\rangle$. To see this, suppose an experimenter prepares the following initial state

$$\begin{aligned}\rho &= \frac{1}{2}(|0\rangle\langle 0| + |N\rangle\langle N|) && (4.17)\\ &= \frac{1}{2}(|0\rangle\langle 0| + \frac{1}{\chi_N N!}(c^\dagger)^N|0\rangle\langle 0|(c)^N. && (4.18)\end{aligned}$$

The above state is an even mixture of the vacuum and the N particle state. Suppose the experimenter performs a counting experiment, then he is equally likely to find 0 particles as N particles. He then adds a particle to the system, resulting in the final state

$$\begin{aligned}\rho' &= c^\dagger\rho c/\mathrm{Tr}(c^\dagger\rho c) && (4.19)\\ &= \frac{1}{(1+(N+1)\frac{\chi_{N+1}}{\chi_N})}\left(|1\rangle\langle 1| + (N+1)\frac{\chi_{N+1}}{\chi_N}|N+1\rangle\langle N+1|\right). && (4.20)\end{aligned}$$

We see from the final state above that the probability of measuring the system in the state $|N+1\rangle$, which we denote p_{N+1}, relative to the probability of finding the state $|1\rangle$, which we denote p_1, is given by

$$p_{N+1}/p_1 = (N+1)\frac{\chi_{N+1}}{\chi_N}. \tag{4.21}$$

Since initially the system is prepared such that you are equally likely to have the vacuum state as the N particle state, but after a successful particle addition, you are more likely to get the $N+1$ particle state than the 1 particle state, it implies that you are more likely to add a composite particle to the N particle state than the vacuum state by the factor $(N+1)\frac{\chi_{N+1}}{\chi_N}$.

Now, suppose we add N particles into N different modes. This is a probabilistic operation, but let us assign a weight of 1 to the success of performing this operation. We now consider the operation of adding all N particles to the *same* mode. The probability of this operation, relative to adding N particles to different modes, is given by

$$(2\frac{\chi_2}{\chi_1})(3\frac{\chi_3}{\chi_2})\cdots(N\frac{\chi_N}{\chi_{N-1}}) = N!\chi_N, \tag{4.22}$$

so we obtain a physical interpretation of the factor χ_N – it is proportional to the probability of success of producing the state $|N\rangle$ from the vacuum state $|0\rangle$ by adding a particle one after another.

4.3 A Semi-Classical Interpretation of χ_N

This section will be devoted towards developing a semi-classical explanation for the factor χ_N. We will see that χ_N may be considered as the amount of degeneracy for a given energy state.

Consider a series of N different 1 dimensional lattices, which we label using the index p. Each point in the optical lattice is labelled by the index n, and at each point, we can place 2 fermions (a bifermion) described by $a^\dagger_{p,n}b^\dagger_{p,n}$. A composite particle is described by the operator $c^\dagger_p$, such that $c^\dagger_p = \sum_n \sqrt{\lambda_n} a^\dagger_{p,n}b^\dagger_{p,n}$.

We assume that the Hamiltonian of the system satisfies, $\hat{H}$ satisfies the following:

$$\hat{H}(c^\dagger_p)^N|0\rangle = NE_p(c^\dagger)^N|0\rangle. \tag{4.23}$$

That is, if there are N composite particles occupying a lattice p, then the energy of the system is simply N times the energy of a single composite particle. For such a system an experimenter may in principle perform a measurement on the occupation number of a particular mode with energy E_p. This measurement is associated with a Hermitian operator which we denote $\hat{N}_p$. Note that the eigenstate of $\hat{H}$ is also an eigenstate of $\hat{N}_p$, so they commute and share the same eigenbasis. As a consequence, a measurement of $\hat{N}_p$ on an eigenstate of the Hamiltonian does not disturb the state of the system.

We will now consider the eigenstate with N composite particle, given by $(c_p^\dagger)^N|0\rangle$. Suppose the experimentalist measures first the occupation number of the mode p. His measurement outcome will him that there are N composite particles, each with energy E_p in his system. Subsequently, he measures the position of each of the N composite particle (described by the quantum number n) by checking for a fermion pair in each of the lattice positions n. This second measurement will allow him to infer the momentum of all N particles up to a sign factor, so he is able to determine the possible phase space coordinates of the system. We denote the possible measurement outcomes of the positional measurement to be $(n_1, n_2, ..., n_N)$.

For a system in contact with a heat bath, each possible state in phase space is attributed an equal *a priori* probability if they have equal energy. However, for the quantum system under consideration, this assumption cannot be valid because the particle in each lattice position has equal energies (which is the result of the initial measurement), and yet the actual probability of finding a fermion pair in each lattice position is unequal. To illustrate this, consider the state of a single composite particle $c^\dagger|0\rangle$. The occupation measurement will allow the experimenter to infer that the composite particle has energy E_p. The probability of finding the particle in the lattice position n is however, given by λ_n, which in general is not equal for all n. This is incompatible with the assumption of equal *a priori* probability for every possible state in phase space, but only if we assume that the position and momentum are all that is necessary to describe the state of the particle.

In order to resolve this, we will introduce some level of degeneracy to each lattice position n. Denoting the degeneracy at each lattice position Ω_n, we define the corresponding degeneracy such that it satisfies the following:

$$\frac{\Omega_n}{\sum_i \Omega_i} = \lambda_n. \tag{4.24}$$

One may think of this extra degeneracy as some classical hidden variable $\mu(n)$, inaccessible to the experimentalist, which ascribes $\Omega_n = \sum_{\mu(n)}$ possible different values for each lattice coordinate n. As a consequence of this, the complete state of the system (up to a sign factor in the momentum) is described by $(\mu(n), n, E_p)$. By attributing each unique state an equal *a priori* probability, then a simple calculation shows that the probability of obtaining coordinate n for a single composite particle is λ_n, as expected.

We may extend this to a system of N composite particles in a relatively straightforward manner. For a quantum system the possible outcomes for the set of measurement coordinates $(n_1, n_2, ..., n_N)$. We will assume that all composite particles are identical, so each measurement outcome is not associated to any particular composite particle. We may be assume that the position measurements are ordered in increasing order such that $n_1 < n_2 < ... < n_N$, since

$$(c_p^\dagger)^N|0\rangle = \sum_{n_1<n_2<...<n_N} N! \prod_{i=1}^{N} \lambda_{n_i}(a^\dagger_{p,n_i} b^\dagger_{p,n_i})|0\rangle. \tag{4.25}$$

This is a consequence that $a^\dagger_{p,n_i}$ and $b^\dagger_{p,n_i}$ are identical fermions that obey Pauli's exclusion principle.

Assuming that $(n_1, n_2, ..., n_N)$ satisfies $n_1 < n_2 < ... < n_N$ as required and that the degeneracy for each coordinate n_i is Ω_{n_i} the total degeneracy for the state described by $(n_1, n_2, ..., n_N)$ is given by $\Omega_{n_1} \times \Omega_{n_N} \times ... \times \Omega_{n_N}$. The total degeneracy over all possible combinations of $(n_1, n_2, ..., n_N)$ is simply

$$\sum_{n_1<n_2<...<n_N} \Omega_{n_1}\Omega_{n_N}...\Omega_{n_N}. \tag{4.26}$$

However, we also have the following relation:

$$\begin{aligned}\chi_N &= \sum_{n_1<n_2<...<n_N} \lambda_{n_1}\lambda_{n_2}...\lambda_{n_N} \\ &\propto \sum_{n_1<n_2<...<n_N} \Omega_{n_1}\Omega_{n_N}...\Omega_{n_N},\end{aligned} \tag{4.27}$$

which is compatible with the assumption of equal *a priori* probabilities. The factor χ_N may therefore be interpreted as a measure of the total degeneracy of a an eigenstate of the system.

4.4 A Szilard Engine With 2 Composite Particles

Suppose we have a Szilard Engine composed of 2 composite particles. This engine is placed in some heat bath in the low temperature limit such that $T \to 0$. The low temperature limit is especially interesting conceptually, since the particles will configure themselves such that they occupy the lowest energy state – the ground state.

For elementary identical particles, the amount of work you can extract is intimately related to the effect of Pauli's exclusion principle. Consider for instance 2 ideal bosons in a box partitioned into 2 symmetric sides. For bosons, you may freely distribute the particles on both sides to achieve the lowest energy configuration. This is not the case for an ideal fermion. Due to Pauli's exclusion principle, no two particles can occupy the same eigenstate on one side of the partition.

The situation for composite particles is more complex, as how closely they relate to a boson or a fermion depends on how the constituents of the particles interact with each other. Unlike elementary fermions, one cannot definitively rule out that 2 composite particles can occupy the same mode so long as the amount of entanglement it contains is non-zero, since $\|(c^\dagger)^2|0\rangle\|^2 \neq 0$, so it is a valid state of the system. On the other hand, one cannot also immediately associate a composite particle with bosons because if the constituents of the composite particle are only very weakly correlated with each other, then $c^\dagger$ is algebraically very similar to a fermionic creation operator, so observing two composite particle occupying the same mode must be a comparatively difficult and rare event. In the following treatment of the Szialrd engine with composite particles, we will make use of the physical interpretation of χ_N, which was discussed in the preceding section.

We will account for the rarity of occupying a certain mode by considering the relative probability of the states under consideration. We first define the relative probability for a given set of occupation numbers to be the following:

$$P_{rel}(\{n_p\}) \equiv (\frac{N!}{\prod_{p'} n_{p'}!}) \prod_p n_p! \chi_{n_p} = N! \prod_p \chi_{n_p}, \tag{4.28}$$

where $\{n_p\}$ is the set of occupation numbers n_p for each mode p and satisfies $\sum_p n_p = N$. Note that the term $(\frac{N!}{\prod_{p'} n_{p'}!})$ is the number of possible operations that can be performed to produce a given configureation $\{n_p\}$ of occupation numbers, by adding one particle to the system at a time. $\prod_p n_p! \chi_{n_p}$ is the relative probability of one such operation compared to adding N particles in N different modes, as given by Equation 4.22. Equation 4.28 is therefore the product of the number of possible operations that will produce the final configuration, together with the relative probability of success of said operations, as prescribed by Equation 4.22. In all, this provides the total (relative) probability that a particular configuration of occupation numbers is produced.

For the Szilard engine with 2 particles, we only need to consider 3 possible states of the system: (a) one particle is on each side of the partition (b) both particles are to the left of the partition (c) both particles are to the right of the partition. We will assume that the energy of the system in all 3 cases are identical and equal to some value E_0, corresponding to the ground state energy of the system. As mentioned, the ground state configurations are the only ones that need to be considered in the low temperature limit. In general, there is a possibility that the energy of configurations (b) and (c) may be slightly larger than that of (a). This is because the constituent particles are composed of fermions respecting Pauli's principle. By placing both of these composite particles in the same well, these fermions may experience exchange forces causing the mean separation between the particles to increase which, depending on the system, may in turn increase the observed energy of the state. We shall assume that this increase in energy due to exchange forces are negligible, and lies within the natural thermal fluctuations of the system. This assumption is valid in the regime where the dimensions of the trapping potential is macroscopic with respect to the microscopic composite particles. Intuitively, a macroscopic trapping potential should see only the composite particles as a whole and not its constituents. Any effect on the energy of the system due to the microscopic internal structure of the particles is therefore negligible.

Suppose the above physical conditions are satisfied, we can then apply Equation 4.28 to (a),(b) and (c). For (a) There are two operations you can perform leading to 1 particle on each side to the partition: you may add one particle to the left side and followed by adding a particle to the right, or vice versa. Each of this operations correspond to a relative probability of 1, so according to Equation 4.28, its relative probability, or the probabilitic "weight" assigned to it, is $P_{rel}(\{1,1\}) = 2$. For states (b) and (c) There is only one operation to perform, which is adding two particles one after another on one side of the partition, as the particles are identical. This operation corresponds to the relative probability $2\chi_2$, so $P_{rel}(\{2,0\}) = P_{rel}(\{0,2\}) = 2\chi_2$. This implies that the probability of finding 2 particles to the right of the partition is simply

$$f_0 = \frac{\chi_2}{1 + 2\chi_2}. \tag{4.29}$$

Thus the amount of extractable work (See Equation 4.15) from the system increases with the factor χ_2. Note that if $\chi_2 = 1$, as would be expected from an ideal boson, then $f_0 = 1/3$, and if $\chi_2 = 0$, as would be expected for ideal fermions, then $f_0 = 0$, so existing results regarding the elementary particles are retrieved. Interestingly, if $\chi_2 = 0.5$, the amount of work extractable from 2 classical distinguishable particles is obtained. This allows us to employ χ_2 as a measure of the bosonic or fermionic nature of composite particles. If $\chi_2 > 0.5$, then it is distinctively closer to being bosonic in nature, and if $\chi_2 < 0.5$, then it is closer to being a fermion. This observation that the factor χ_2 is deeply related to the Bosonic and Fermionic properties of composite particles has been explored in the previous chapters (See also Refs. [9, 18, 37, 38]) and it may be interpreted as a measure of the amount of entanglement within a single composite particle. Through

Eqns. 4.15 and 4.29, it now gains an additional operational interpretation in terms of the amount of extractable work from a Szilard engine.

4.5 Generalization to N composite particles and general temperature T

In this section, we generalize the above procedure for a Szilard Engine with N composite particles and general temperature T. Note that the amount of extractable work as given in Equation 4.12, is defined entirely by the partition functions of the system, generally given by $Z \equiv \sum_n e^{-\beta E_n}$ where β is the inverse temperature $\frac{1}{k_B T}$, E_n is the energy of the nth state of the system and k_B is the Boltzmann constant. For the Szilard Engine, we may denote a particular state of the system by $(l, \{m_p\}_L, \{n_q\}_R)$, which is defined by the position of the partition l, and the set of occupation numbers $\{m_p\}_L$ and $\{n_q\}_R$ on the left and right of the partition respectively. The corresponding energy of the state is then $E(l, \{m_p\}_L, \{n_q\}_R)$. For elementary particles, one may then expect that the probability of finding the system in the state is proportional to $\exp\left(-\beta E(l, \{m_p\}_L, \{n_q\}_R)\right)$.

However, as per our previous conclusion, for composite particles, the probability of a given state of the system is also related to the set of occupation numbers of the state (See Equation 4.28). This implies that the probability of a given state $(l, \{n_p\}_L, \{m_q\}_R)$ with occupation numbers $\{m_p\}_L$ and $\{n_q\}_R$ is also proportional to the factor: $\prod_p \chi_{m_p} \prod_q \chi_{n_q}$. Thus, the expected probability of a state of the system should $p(l, \{m_p\}_L, \{n_q\}_R)$ satisfies the following:

$$
\begin{aligned}
&p(l, \{n_p\}_L, \{m_q\}_R) \\
&\quad \propto \prod_p \chi_{n_p} \prod_q \chi_{m_q} \exp\left(-\beta E(l, \{n_p\}_L, \{m_q\}_R)\right) \\
&\quad \equiv Z(l, \{m_q\}_L, \{n_p\}_R).
\end{aligned}
\tag{4.30}
$$

This allows us to redefine the necessary partition functions to calculate the extractable work from composite particles. For a given position of the partition l, the partition function of the system with m composite particles to the left of the partition , $Z_m(l)$, is given by

$$Z_m(l) = \sum_{(\{m_q\}_L, \{n_p\}_R) \in M} Z(l, \{m_q\}_L, \{n_p\}_R), \tag{4.31}$$

where the set M is defined to be the set of occupation numbers $(\{m_q\}_L, \{n_p\}_R)$ satisfying $\sum_q m_q = m$ and $\sum_p m_p = N - m$. The total partition function $Z(l)$ is then further given by

$$Z(l) = \sum_{m=0}^{N} Z_m(l), \tag{4.32}$$

where the partition function $Z(l)$ in general depends on the system being studied. With the expressions given by Equations 4.31 and 4.32, it is then in principle possible to compute f_m and f_m^* which, together with the temperature of the bath, fully defines the amount of extractable work for a Szilard engine composed of N composite particles at temperature T, as given by Equation 4.12. Note that $1 - N(1 - \chi_2) \leq \frac{\chi_{N+1}}{\chi_N} \leq \chi_2$ [18], and as a result the partition functions for bosons and fermions are retrieved in the limits $\chi_2 \to 1$ and $\chi_2 \to 0$ respectively.

4.6 Chapter Summary

In this chapter, we discussed the Szilard engine. The Szilard engine is a hypothetical engine producing work, where the primary resource consumed in order to produce this work is *information*. This relationship between information and work extraction is an interesting topic in itself, but the focus here is not on this relationship. Rather, we discuss how the amount of work extracted may be related to the amount of entanglement contained in the composite particle.

In order to perform this analysis, we considered the physical interpretation of the quantity χ_N. In particular, in a semi-classical argument, we demonstrate that the χ_N may be considered as a measure of degeneracy of a particular eigenstate. Based on such an interpretation, we are able to formally write down the partition function of a system of composite particles, obtaining an expression for the work done by a Szilard engine in terms of the factor χ_N.

Of particular interest is the 2 particle case. Just as bosons and fermions are concepts that emerge when the system has 2 or more particles (it is only then that we can talk about indistinguishability), the "boson-ness" and "fermion-ness" of composite particles only reveal themselves when there are 2 or more composite particles in the Szilard engine. In a 2 particle case, we have a particularly concise expression (See Eqn. (4.29)) for the amount of work extracted from the engine in the low temperature limit. This quantity depends only on χ_2, and is independent of any other physical parameter such as the energy structure. As such, we can use it to define an alternative measure of the bosonic quality of the composite particle.

Chapter 5

Conclusion and Summary

In this thesis, we discussed the role that entanglement plays in composite particles. The focus is primarily on composite particles composed of 2 distinguishable particles, especially composite particles of 2 fermions. The main reason is that entanglement as a resource in bipartite situations is more well developed and understood than entanglement between 3 or more systems. Nonetheless, there are some proposals for quantifying entanglement in larger systems (See [39, 40]). Their relation to the physical properties of composite particles remain at this point unclear, however. The exploration of more general, multipartite entanglement and their role in composite particles is one that warrants further investigation.

In Chapter 1, we begin the discussion by introducing the operators that describe bosons and fermions, before proving some known results regarding composite particles that are systems of correlated but distinguishable fermions. We also briefly discuss the quantification of entanglement, where it is to be viewed as a resource quantity, and provide a proof that entanglement allows fermion pairs to adopt properties of an ideal boson operator.

In Chapter 2 we introduced the concept of *LOCC condensation*, where we define condensation to be a final state where many identical composite particles occupy the same state. The key motivation behind this is to study the role that entanglement plays in the condensation process. By limiting the operations that may be performed on composite particles to only locally performable quantum operations, possibly supplemented with classical communication, we remove the role that interaction plays in the condensation process, leaving behind quantum correlation as the only resource for condensation. We demonstrate that even strongly correlated composite particles may not be sufficient to LOCC condense, and that in fact there exists systems that are only infinitesimally correlated, and yet are able to LOCC condense in spite of this weak correlation. As a consequence, we show that entanglement is necessary for condensation, but not sufficient in order for condensation to always be possible.

In Chapter 3, we propose a way to measure the bosonic quality of composite particles using the elementary operation of particle addition and subtraction. Our proposed measure, due to its elementary nature, is in fact applicable for all particle types. An analysis of different particle types allow us to conclude that entanglement is not a resource for producing boson like particles. While it is true that fermion pairs increasingly behave like bosons the more entangled they are, boson pairs become *less* bosonic in the sense that they behave less like a pair of bosons, and increasingly behave more like a *single* boson. The true character of entanglement is therefore suggested to be a resource that makes many particle systems behave more like a single particle, rather than a resource that increases bosonic effects. We also directly relate our measure to a particle bunching/anti-bunching experiment.

In Chapter 4, we introduced the *Szilard engine*, an engine whose primary resource to extract work is information. We first discuss how a quantity χ_N, related to the entanglement of the system, may be interpreted as the degeneracy of a given energy eigenstate. Following this line of inquiry, we were able to demonstrate that a Szilard engine using composite particles composed of 2 distinguishable fermions is able to extract more work when entanglement is high.

As a conclusion to this thesis, it is worth mentioning that there remains many other possible interesting question one may ask regarding composite particles. For instance, an important issue is how to generalize the existing results we have on pairs of particles to general, larger systems. A general relationship between entanglement and bosonic quality does not exist for larger systems of fermions currently do not exist. We may also ask whether there is any role that entanglement in a composite particle plays in the context of a Carnot engines (as opposed to the Szilard engine discussed here), or even quantum refridgerators, given that entanglement in fermion pairs directly affects the partition function. In light of the discussion regarding the quantum Szilard engine, where Maxwell's demon uses information to extract work, it may also be worthwhile to investigate how composite particles may lead to interesting physics in Maxwell's original thought experiment, where Maxwell's demon uses information to cool a system. We are hopeful that some of these questions will be answered in the near future.

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