COLLÈGE DE FRANCE 1530

ECOLE NORMALE SUPERIEURE

Collège de France abroad Lectures

Quantum information with real or artificial atoms and photons in cavities

Lecture 2:

A review of quantum measurement theory illustrated by the description of QND photon counting in Cavity QED

NUS National University of Singapore

II-A

A review of measurement theory: standard, POVM and generalized measurements

Standard measurement

A standard (von Neumann) measurement is determined by giving the ensemble of projectors on a basis of eingenstates of an observable G (represented by a hermitian operator), for which a measuring apparatus (or meter) has been defined: .

$$G|a_i\rangle = g_i|a_i\rangle \quad ; \quad \langle a_i|a_j\rangle = \delta_{ij}$$

$$\Pi_i = |a_i\rangle\langle a_i| \quad ; \quad \sum_i \Pi_i = I \quad ; \quad \Pi_i\Pi_j = \delta_{ij}\Pi_i$$

The probability p_i for finding the result i on a system in the state ρ before measurement and the state ρ_i in which the system is projected by the measurement are given by:

$$p_i = Tr\rho\Pi_i \; ; \sum_i p_i = 1$$

$$\rho \rightarrow \rho_i = \frac{\Pi_i \rho \Pi_i}{p_i}$$

The number of projectors is equal to the Hilbert space dimension (2 for a qubit) and the measurement is repeatable: After a first measurement, one finds again the same result, as a direct consequence of the orthogonality of the projectors Π_i.

Besides these standard measurements (also called projective measurements), one can define also generalized or POVM measurements (for Positive Operator Valued Measure). See next page.

POVM measurement

A POVM is defined by an ensemble of positive hermitian operators E_i (having non-negative expectation values in all states) realizing a partition of unity:

$$\sum_i E_i = I$$

The number of E_i operators can be arbitrary, either smaller or larger than the Hilbert space dimension. The POVM is defined by the rules giving the probabilities p_i for finding the result i and the state after measurement, which generalize standard measurement rules:

$$p_i = Tr\{\rho E_i\} \quad ; \quad \sum_i p_i = 1$$

$$\rho \rightarrow \rho_i = \frac{\sqrt{E_i}\rho\sqrt{E_i}}{p_i}$$

Since the E_i are not normalized projectors, one has in general:

$$E_i E_j \neq 0 \; if \; i \neq j \quad ; \quad E_i^2 \neq E_i$$

The POVM process is a statistical measurement since its yields a result belonging to a set of values, with a probability distribution. The measurement is not repeatable (with mutually exclusive results). One can find different results successively when resuming the measurement.

POVM realized as a standard measurement on an auxiliary system

A POVM on a system S in a Hilbert space (A) can always be reduced to a projective measurement in an auxiliary system belonging to another space (B), to which S is entangled by a unitary transformation. Let us associate to each element i of the POVM a vector $|b_i\rangle$ of B, the $|b_i\rangle$'s forming an orthonormal basis (space B has a dimension at least equal to the number of POVM elements) and let us consider a unitary operation acting in the following way on a state $|\psi\rangle_A |0\rangle_B$, tensor product of an arbitrary state of A with a « reference » state $|0\rangle_B$ of B:

$$|\psi\rangle_A |0\rangle_B \rightarrow \sum_i \sqrt{E_i}\, |\psi\rangle_A |b_i\rangle_B$$

This operation conserves scalar products and is thus a restriction of a unitary transformation in (A+B). Applied to a statistical mixture of (A), it writes by linearity:

$$\rho_A \otimes |0\rangle_B {}_B\langle 0| \rightarrow \sum_{i,j} \sqrt{E_i}\, \rho_A \sqrt{E_j} \otimes |b_i\rangle_B {}_B\langle b_j|$$

Let us then perform a standard measurement of an observable of B admitting the $|b_i\rangle$'s as eigenstates. The result i is obtained with the probability of the POVM and the system S is projected according to the POVM rule. We have realized in this way the desired POVM. Two element-POVM's can thus be realized by coupling S to a qubit, then measuring the qubit, as discussed in next pages.

Generalized measurement

A generalized measurement $\mathcal{M}$ is defined by considering a set of (non necessarily Hermitian) operators M_i of a system A fulfilling the normalization condition:

$$\sum_i M_i^\dagger M_i = I$$

The result i of the measurement $\mathcal{M}$ occurs with the probability:

$$p_i = Tr\left\{ M_i \rho M_i^\dagger \right\}$$

and the system after measurement is projected onto the state:

$$\rho_{proj(i)} = \frac{M_i \rho M_i^\dagger}{p_i}$$

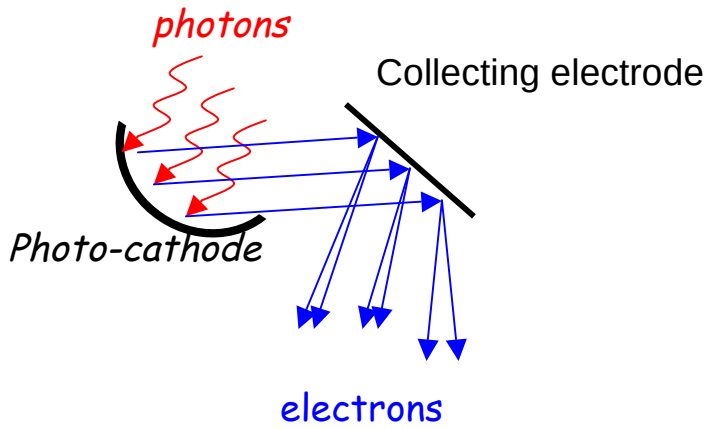
The generalized measurement $\mathcal{M}$ can be realized by coupling A to an auxiliary system B by the unitary defined as:

$$U_M \left|\psi^{(A)}\right\rangle \otimes \left|0^{(B)}\right\rangle = \sum_i M_i \left|\psi^{(A)}\right\rangle \otimes \left|u_i^{(B)}\right\rangle$$

where the $|u_i^{(B)}\rangle$ form an orthonormal set of states of B. A standard measurement of B admitting the $|u_i^{(B)}\rangle$ as eigenstates realizes the generalized measurement on A. The POVMs are obviously a special case of generalized measurements with $M_i = M_i^\dagger = \sqrt{E_i}$.

Photo-detection as generalized measurement

Consider a detector able to resolve photon numbers by absorbing the photons of a field mode and yielding a photo-current proportional to n. It corresponds to a generalized measurement with the M_n operators defined as:



$$M_n = |0\rangle\langle n|$$

which obviously satisfy the closure relationship $\Sigma M_n^\dagger M_n = I$. Performing the generalized measurement yields the result n with probability:

$$p(n) = Tr\left\{ M_n \rho M_n^\dagger \right\} = \langle n|\rho|n\rangle$$

and the fields ends (for any number n of photons) in the final state:

$$\rho_{proj} = M_n \rho M_n^\dagger / p(n) = |0\rangle\langle 0|$$

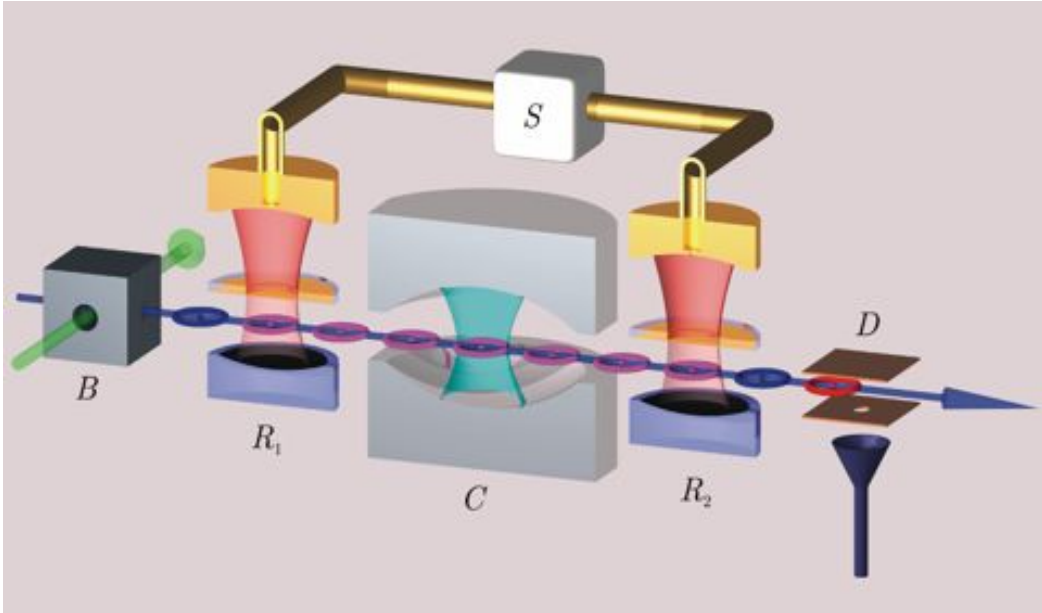
This generalized measurement is a destructive process which always leaves the field in vacuum. Note that this is not a standard measurement which should leave the field in the eigenstate |n> after the result n has been found (see later). A standard measurement must be non-destructive (Quantum Non-demolition).

II-B

Measuring a non-resonant atom by Ramsey interferometry realizes a binary POVM of the field in a Cavity QED experiment

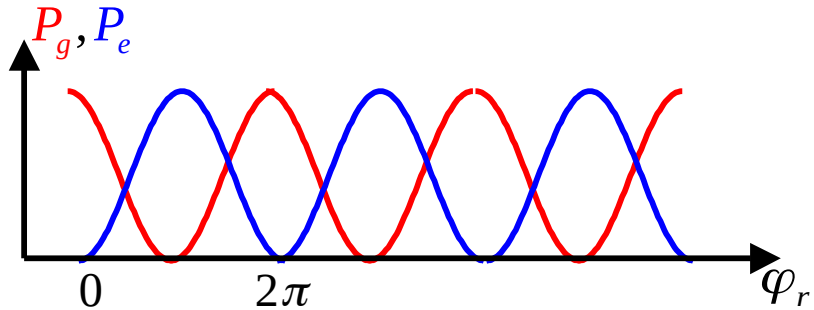
A reminder about Ramsey Interferometer

Let us consider the Ramsey interferometer with the two cavities R_1 et R_2 sandwiching the cavity *C* containing the field to be measured. The atom with two levels *g* and *e* (qubit in states j=0 and j=1 respectively), prepared in *e*, is submitted to classical $\pi/2$ pulses in R_1 and R_2, the second having a φ_r phase difference with the first. The probabilities to detect the atom in *g* (j=0) and *e* (j=1) when *C* is empty are:



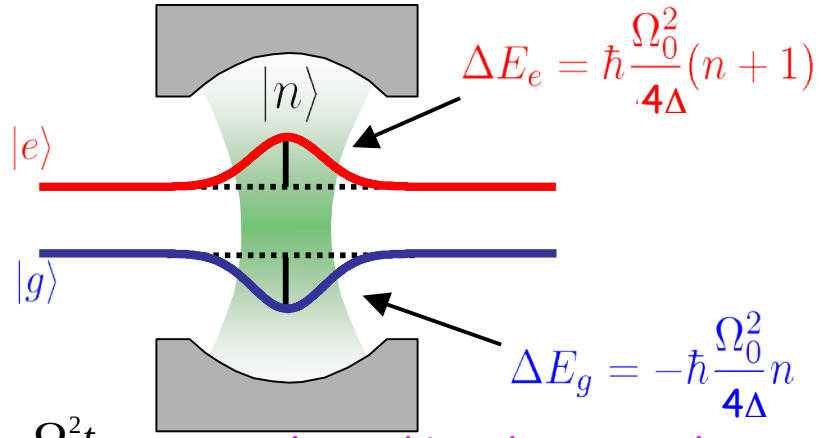
$$P_j = \cos^2 \frac{(\varphi_r - j\pi)}{2} \quad ;$$

The P_j probabilities oscillate ideally between 0 and 1 with opposite phases when φ_r is swept (Ramsey fringes).



A 2-element field POVM realized with an atom qubit dispersively coupled to the cavity

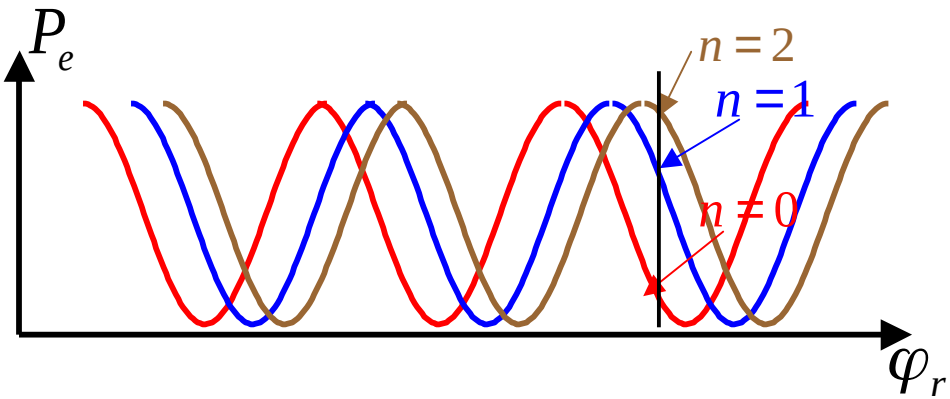
If C is non-resonant with the atomic transition (detuning δ) and contains n photons, the atomic dipole undergoes in C a phase-shift $\Phi(n)$, function of n, linear in n at lowest order (see above). The fringes are shifted and the P_j probabilities become:



$$P_j(n) = \cos^2 \frac{(\varphi_r + \Phi(n) - j\pi)}{2} \quad ; \quad \Phi(n) = \phi_0 n + O(n^2) \quad ; \quad \phi_0 = \frac{\Omega_0^2 t_{eff}}{2\Delta}$$

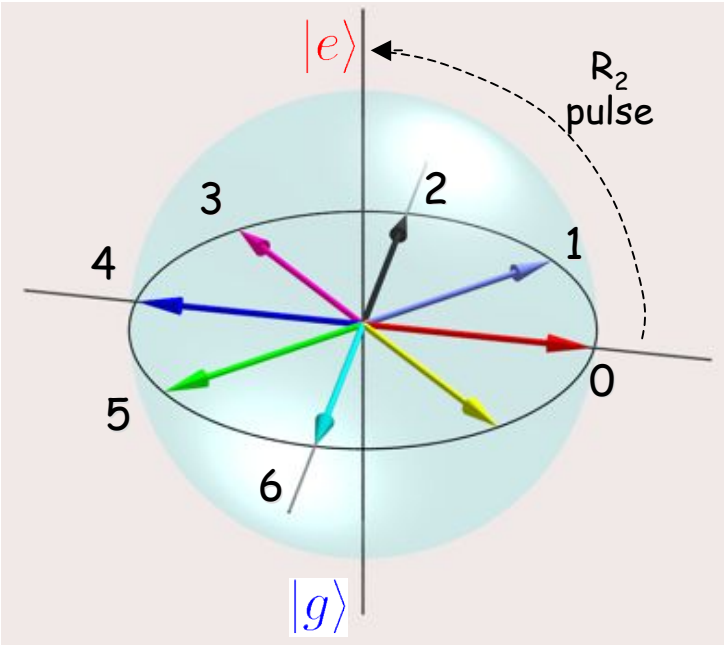
ϕ_0 (tuned by changing Δ) can reach the value π

We have defined the phase-shift per photon ϕ_0, proportional to t_{eff}, effective cavity crossing time taking into account the spatial variation of the coupling. The fringe phase shift allows us to measure the photon number in a non-destructive way (QND method). Each atomic detection realizes a two-element POVM (see next page).



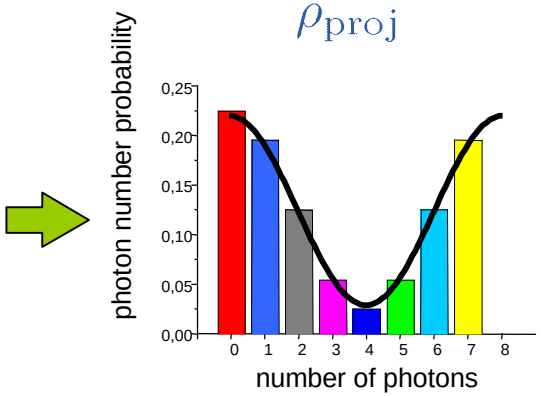
For a given phase φ_r, the probability for finding the atom in e (or g) takes different n-dependent values

Information obtained by detecting 1 atom



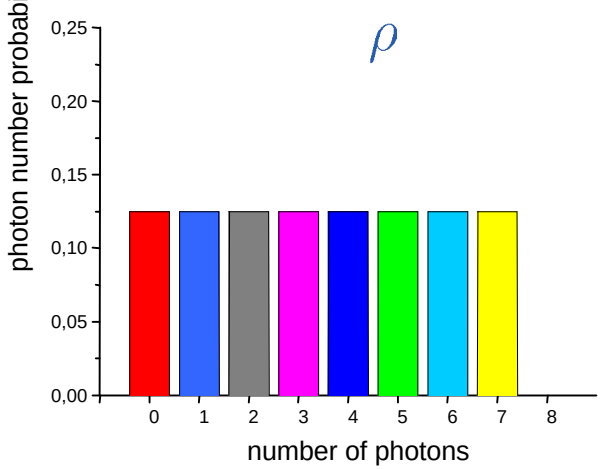
Detecting the Ramsey signal with phase φ_r amounts to chosing a detection direction of the qubit Bloch vector in the equatorial plane of the Bloch sphere. The phase-shift per photon $\phi_0 = \pi/4$ is set to distinguish photon numbers (from 0 to 7), each one corresponding to a different direction of the Bloch vector.

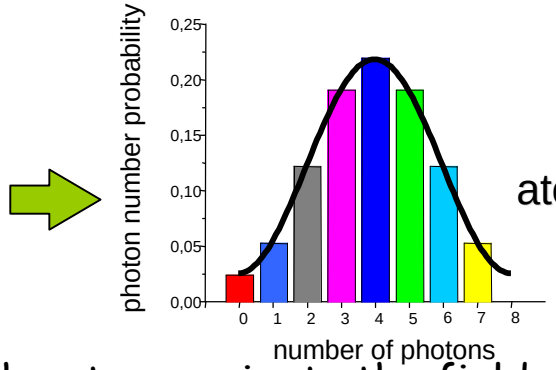
detection direction



atom in $|e\rangle$

A priori no information on n





atom in $|g\rangle$

Measuring the atom projects the field density operator by modulating the $\Pi(n)$ probability with a sine-term whose phase is given by the Ramsey fringe (Bayes law).

LII. *An Essay towards solving a Problem in the Doctrine of Chances. By the late Rev. Mr. Bayes, communicated by Mr. Price, in a letter to John Canton, M. A. and F. R. S.*

A B

a_i b_j

$p(a_i)$ $p(b_j)$

$$p(a_i, b_j) \neq p(a_i)p(b_j) \quad \textit{correlations}$$

$$p(a_i, b_j) = p(a_i \mid b_j)p(b_j) = p(b_j \mid a_i)p(a_i)$$

$\longrightarrow$

$$\boxed{p(b_j \mid a_i) = p(a_i \mid b_j)\frac{p(b_j)}{p(a_i)} = p(a_i \mid b_j)\frac{p(b_j)}{\sum_j p(a_i \mid b_j)p(b_j)}}$$

Bayes law or projection postulate?

The conditional probability to detect the atom in state j (0 or 1) provided they are n photons in *C* is:

$$p(j\,|\,n) = \cos^2\left(\frac{\varphi_r + \Phi(n) - j\pi}{2}\right)$$

The reciprocal conditional probability to have n photons in *C* provided that the atom has been detected in j is given by Bayes law:

$$p(n\,|\,j) = \frac{p(j\,|\,n)\Pi(n)}{p(j)} = \frac{p(j\,|\,n)\Pi(n)}{\sum_n p(j\,|\,n)\Pi(n)}$$

Within normalization, the inferred photon number probability is the a priori one $\Pi(n)$ multiplied by the Ramsey fringe function. The same result is obtained by applying the projection postulate to the qubit measurement. After crossing the Ramsey interferometer, the field (initially in state $\Sigma_n C_n|n\rangle$) and the qubit (initially in e) end up in the entangled state:

$$\sum_n C_n|n\rangle\otimes|j=1\rangle \xrightarrow[Ramsey]{} \sum_n C_n \sin\frac{\varphi_r+\Phi(n)}{2}|n\rangle\otimes|j=1\rangle + C_n\cos\frac{\varphi_r+\Phi(n)}{2}|n\rangle\otimes|j=0\rangle$$

whose projection, conditioned to finding the result j, leads to Bayes formula for the probability for finding n photons in *C*.

Bayes law and the projection rule yield identical results.

The dispersively coupled single atom measurement is a two element POVM

Let us define the two field operators, hermitian and positive:

$$E_j = \cos^2 \frac{\left(\varphi_r + \Phi(a^\dagger a) - j\pi\right)}{2}$$

which satisfy the closure relationship:

$$\sum_j E_j = E_0 + E_1 = I$$

The E_j form a two-element POVM realized by detecting the atom. If the field is initially described by the density operator ρ, the probability for finding the result j is indeed:

$$P_j(\rho) = \sum_n \rho_{nn} P_j(n) = \sum_n \rho_{nn} \cos^2 \frac{\left(\varphi_r + \Phi(n) - j\pi\right)}{2} = Tr\{\rho E_j\}$$

and after atomic detection, the field is projected in state:

$$\rho_{proj}(j) = \sum_{n,n'} \frac{\cos\left[\frac{\varphi_r + \Phi(n) - j\pi}{2}\right] |n\rangle \rho_{n,n'} \langle n'| \cos\left[\frac{\varphi_r + \Phi(n') - j\pi}{2}\right]}{P_j(\rho)} = \frac{\sqrt{E_j}\rho\sqrt{E_j}}{Tr\{\rho E_j\}}$$

These equations define a POVM which realizes a partial measurement of the photon number N. As we will now show, a sequence of such POVM's realizes a standard projective measurement of the photon number, which is of the QND type.

The QND POVM: essential formulae

State before measurement

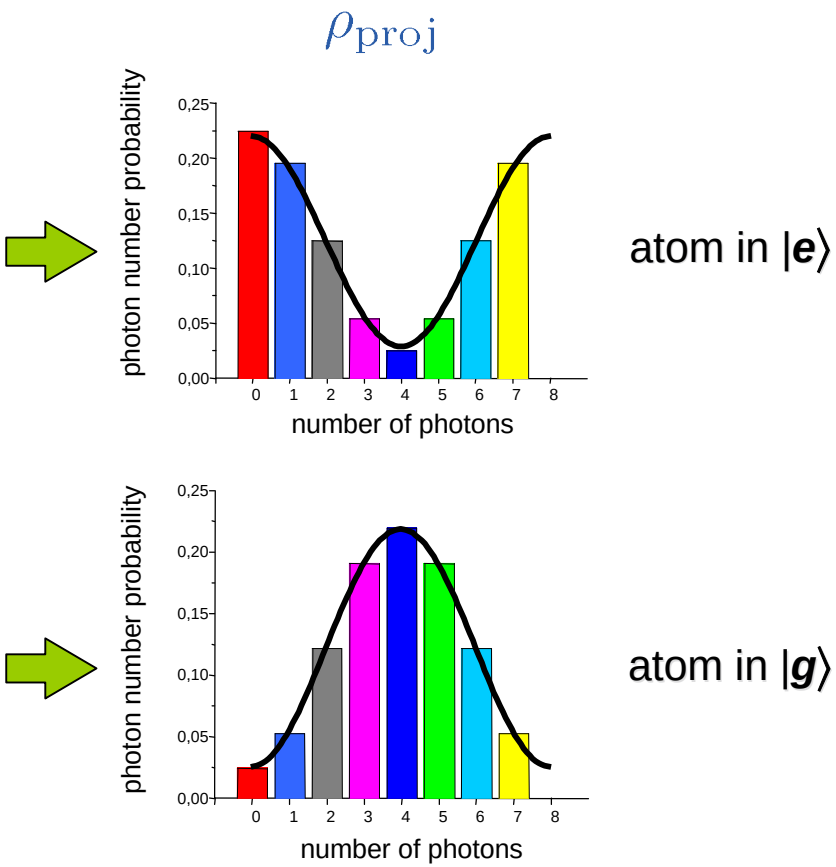
$$\rho_{proj} = \frac{\sqrt{E_j}\,\rho\sqrt{E_j}}{Tr\left(\rho E_j\right)}$$

Projected state

$$E_e = \sin^2\left(\frac{\phi_r + \Phi(N)}{2}\right)$$

The 2 elements of the POVM corresponding to the two possible results (e or g)

$$E_g = \cos^2\left(\frac{\phi_r + \Phi(N)}{2}\right)$$



When realizing a sequence of measurements, each yields a new state. The sequence converges towards a well defined photon number: progressive collapse leading to a standard measurement.

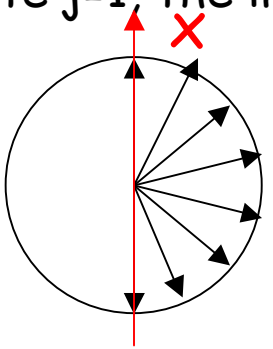
Detecting the atom changes the inferred photon number distribution (more generally the field density operator)

II-C

Quantum Non-Demolition measurement of photon number in a cavity: progressive collapse of the field state

Sequence of POVM's realizing a QND measurement of photon number

To count up to n_m photons, we can either chose $\varphi_0=\pi/(n_m+1)$ and use one detection phase φ_r (corresponding for example to the detection of σ_x) or chose $\varphi_0=2\pi/(n_m+1)$ and use two detection phases (corresponding to the detection of σ_x and σ_y). Let us consider here the first setting. After detecting p qubits in state j=0 and N-p in state j=1, the inferred photon number distribution has become:



$$p(n\,|\,p;N-p)\approx\cos^{2p}\left(\frac{\varphi_r+n\varphi_0}{2}\right)\sin^{2(N-p)}\left(\frac{\varphi_r+n\varphi_0}{2}\right)$$

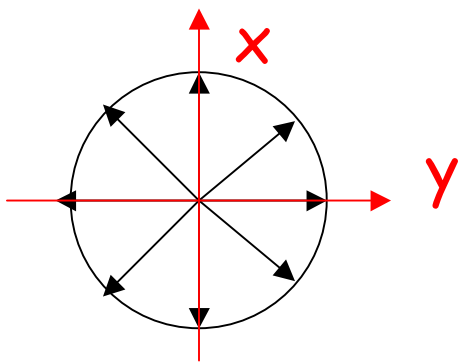
$$=X^p(u)\left[1-X(u)\right]^{N-p}\quad with\quad X(u)=\cos^2 u\ and\ u=\left(\varphi_r+n\varphi_0\right)/2$$

The distribution maximum is obtained by computing the derivative of p(n|p;N-p) versus n. A simple calculation yields:

$$\frac{dp(n\,|\,p,N-p)}{dn}=A\times\left[p-NX\right]\quad with\quad A=-\varphi_0\sin u\cos u\ X^{p-1}\left[1-X\right]^{N-p-1}$$

The derivative cancels for X=p/N and the photon number n_{max} satisfies:

$$X_{\max}=\cos^2\left(\frac{\varphi_r+n_{\max}\varphi_0}{2}\right)=\frac{p}{N}\quad\Rightarrow\quad n_{\max}=\frac{2}{\varphi_0}Arc\cos\left[\sqrt{\frac{p}{N}}\right]-\frac{\varphi_r}{\varphi_0}$$



Two settings for counting up to 7 photons

Sequence of POVM's realizing a QND measurement of photon number (cont'd)

To estimate the width of the inferred photon number distribution we compute its second derivative at X=p/N:

$$\left.\frac{d^2 p(n\,|\,p,N-p)}{dn^2}\right|_{\left(X=\frac{p}{N}\right)} = A\times\frac{d}{dn}\left[p-NX\right] = -A\times N\frac{dX}{du}\frac{du}{dn} = A\times N\varphi_0\sin u\cos u = -N\varphi_0^2 X^p\left[1-X\right]^{N-p} = -N\varphi_0^2 p(n_{\max}\,|\,p,N-p)$$

and we get the Taylor expansion of the photon number distribution around its maximum:

$$p(n\,|\,p,N-p) = p(n_{\max}\,|\,p,N-p)\left[1-\frac{N\varphi_0^2}{2}(n-n_{\max})^2\right] \approx p(n_{\max}\,|\,p,N-p)\,e^{-\frac{N\varphi_0^2}{2}(n-n_{\max})^2}$$

It shows that the distribution is quasi-gaussian with a half-width:

$$\delta n \approx \frac{1}{\varphi_0\sqrt{N}}$$

A precise photon number is pinned down when $\delta n < 1$, or:

$$N > \frac{1}{\varphi_0^2} \approx \frac{\left(n_m+1\right)^2}{\pi^2}$$

When information is extracted independently by the N qubits, their number must be of the order of the square of the dimension of the photon number Hilbert space. More efficient strategies in which the POVM's are adapted to the results of previous measurement allow for a faster convergence.

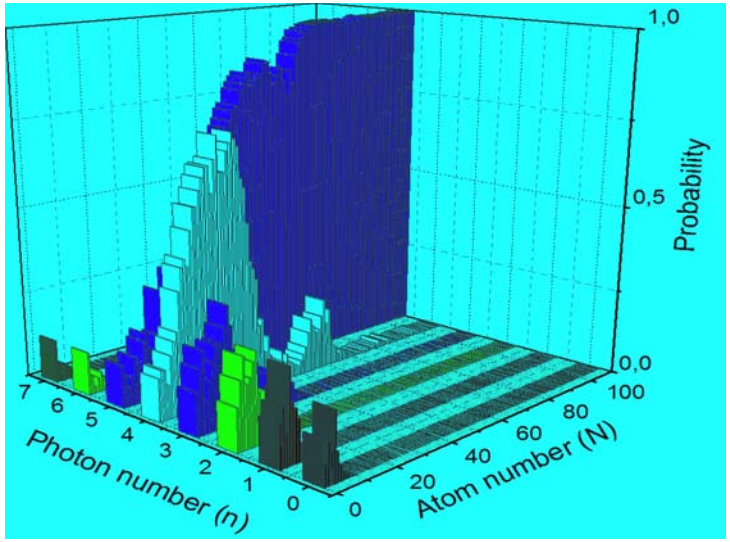
Progressive collapse as n is pinned down to one value

Which number will win the race?

Bayes law in action…

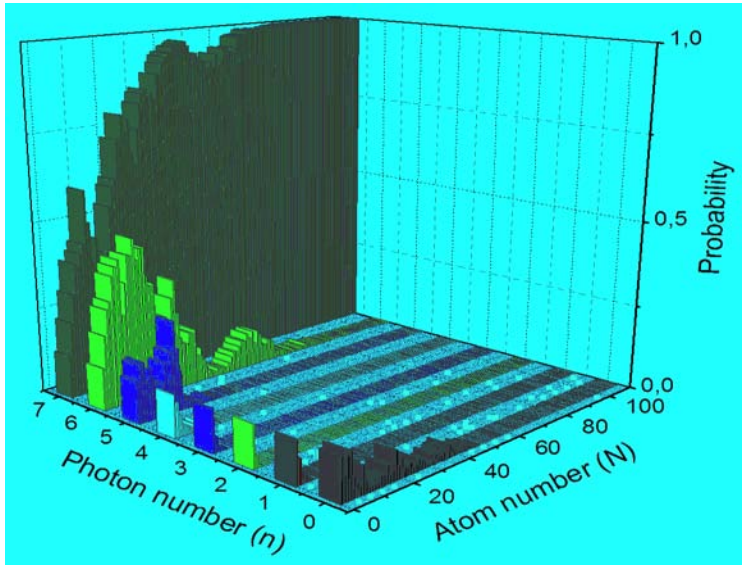
$n = 7 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0$

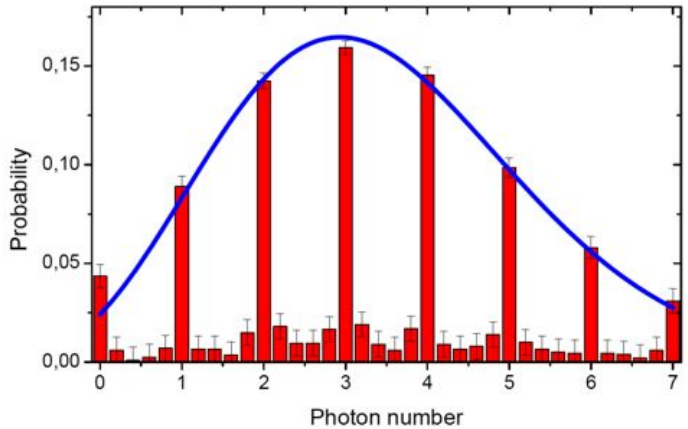
Experimental results



At left: evolution of the inferred photon number distribution in two sequences of ~100 POVM measurements on an initial coherent state with an average of 3,7 photons. The first measurement converges towards n=5 and the second towards n=7 (the successive POVM's correspond to 4 phases ϕ_r alternatively chosen). Probability distributions inferences take into account apparatus imperfections (see later)

At bottom: histogram of measurement results reconstructed from 2000 trajectories; the Poisson law is recovered.

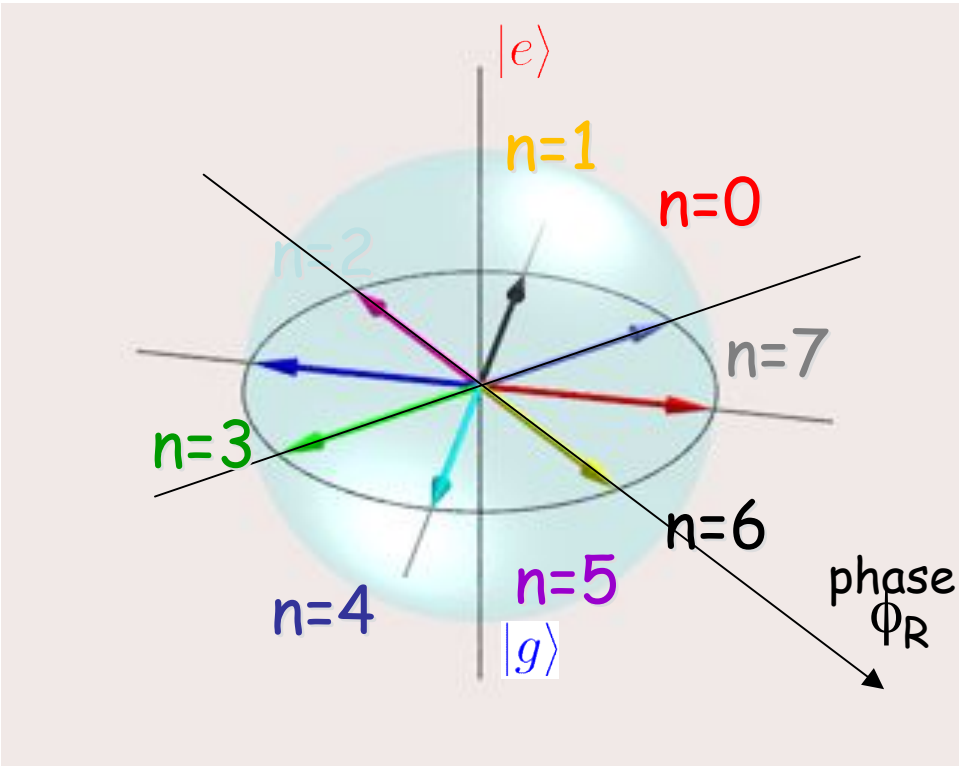


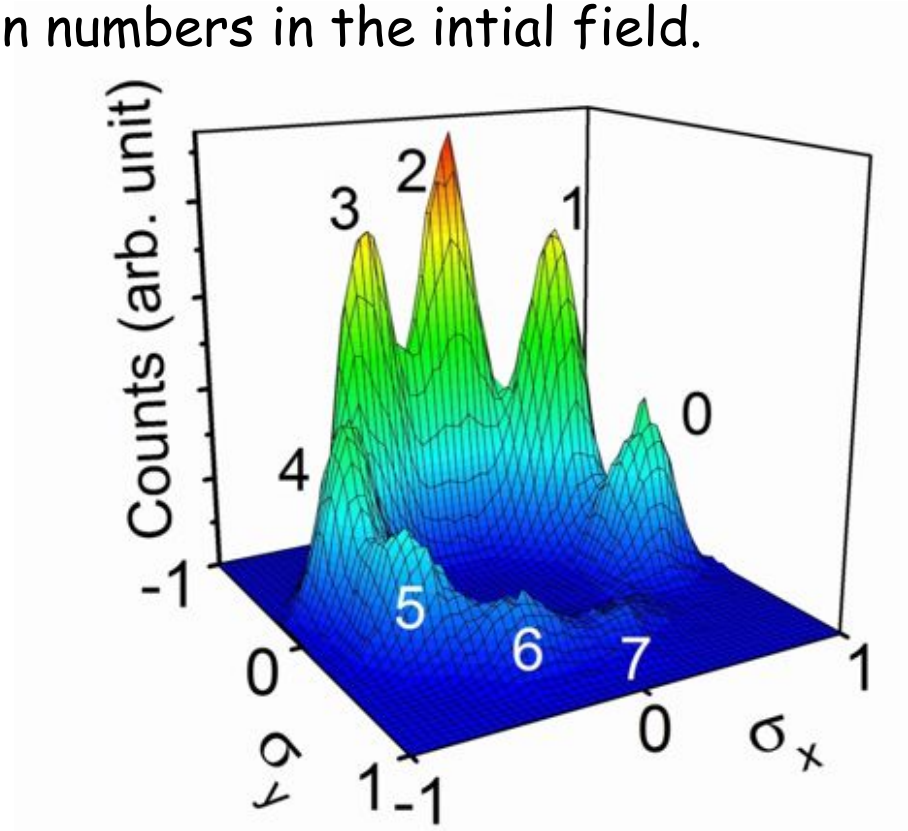


The measurement randomly prepares Fock states. Is it possible to modify the procedure to drive the result towards a preselected Fock state? Quantum feedback- see later.

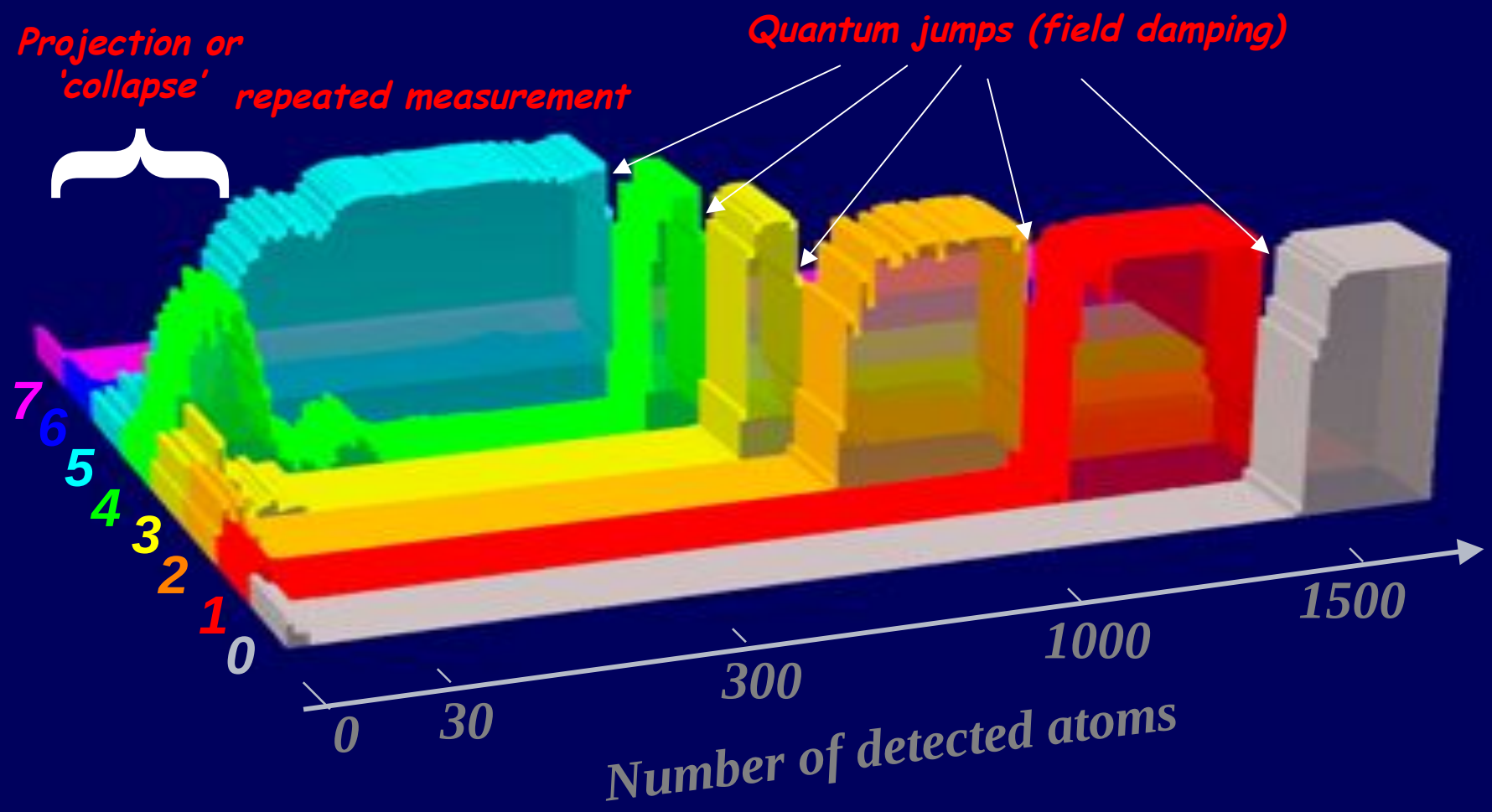
Reconstructing Π(n): an equivalent picture

Instead of computing the product of cosine functions converging towards a delta function, one can equivalently perform a tomographic measurement of N qubits having interacted with the field in a time short compared with the cavity decoherence time. Each sequence yields an expectation value of σ_x and σ_y, hence a vector in the equatorial plane of the Bloch sphere associated to an n value. By resuming the sequence a large number of times, we reconsruct an histogram yielding the distribution of Bloch vector directions, i.e. the distribution of photon numbers in the intial field.



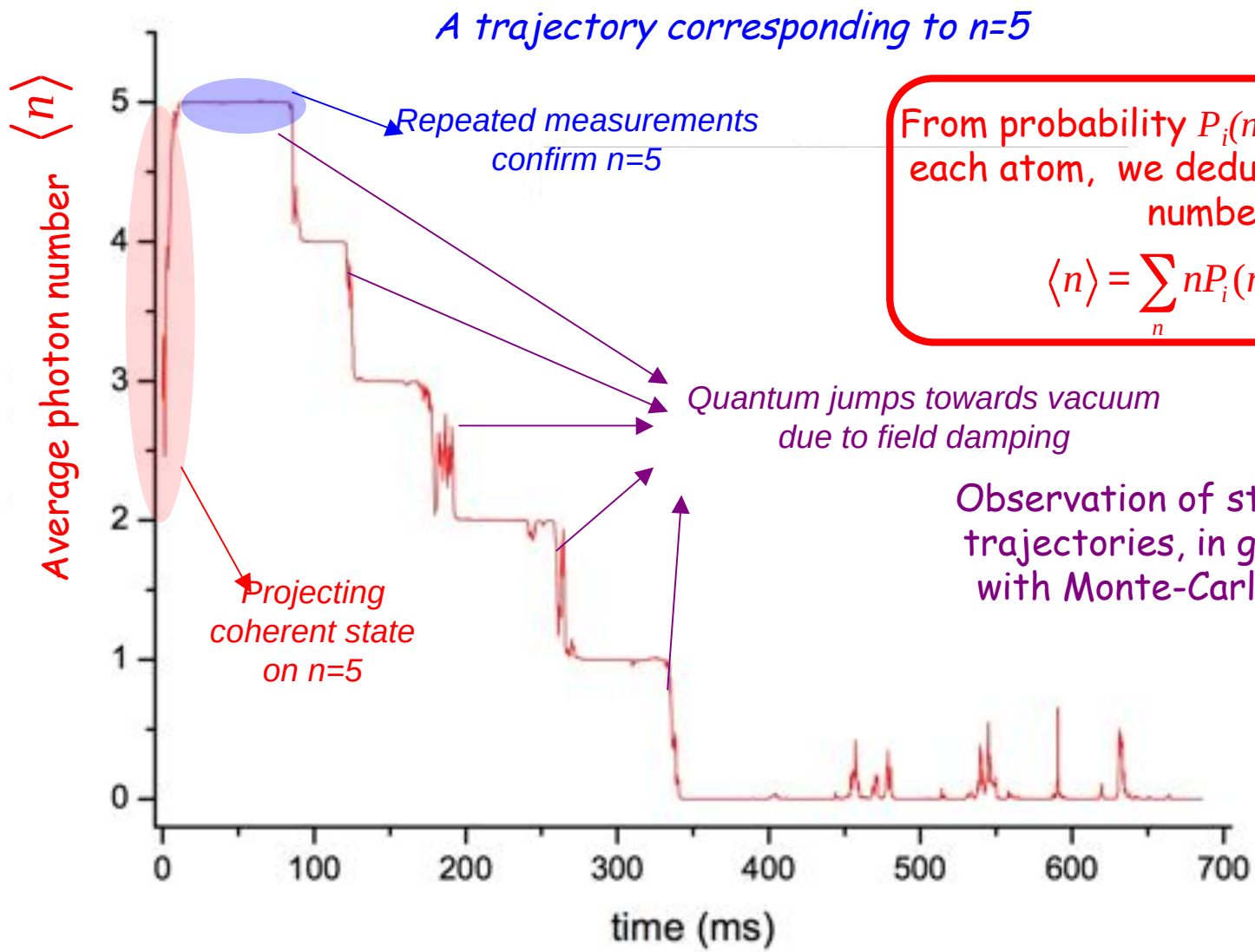


Evolution of the probability distribution over a long sequence: quantum jumps



Distribution $P_i(n)$ obtained with a swept sample of 110 atoms.

Expectation value of photon number along a long POVM sequence: stochastic trajectory



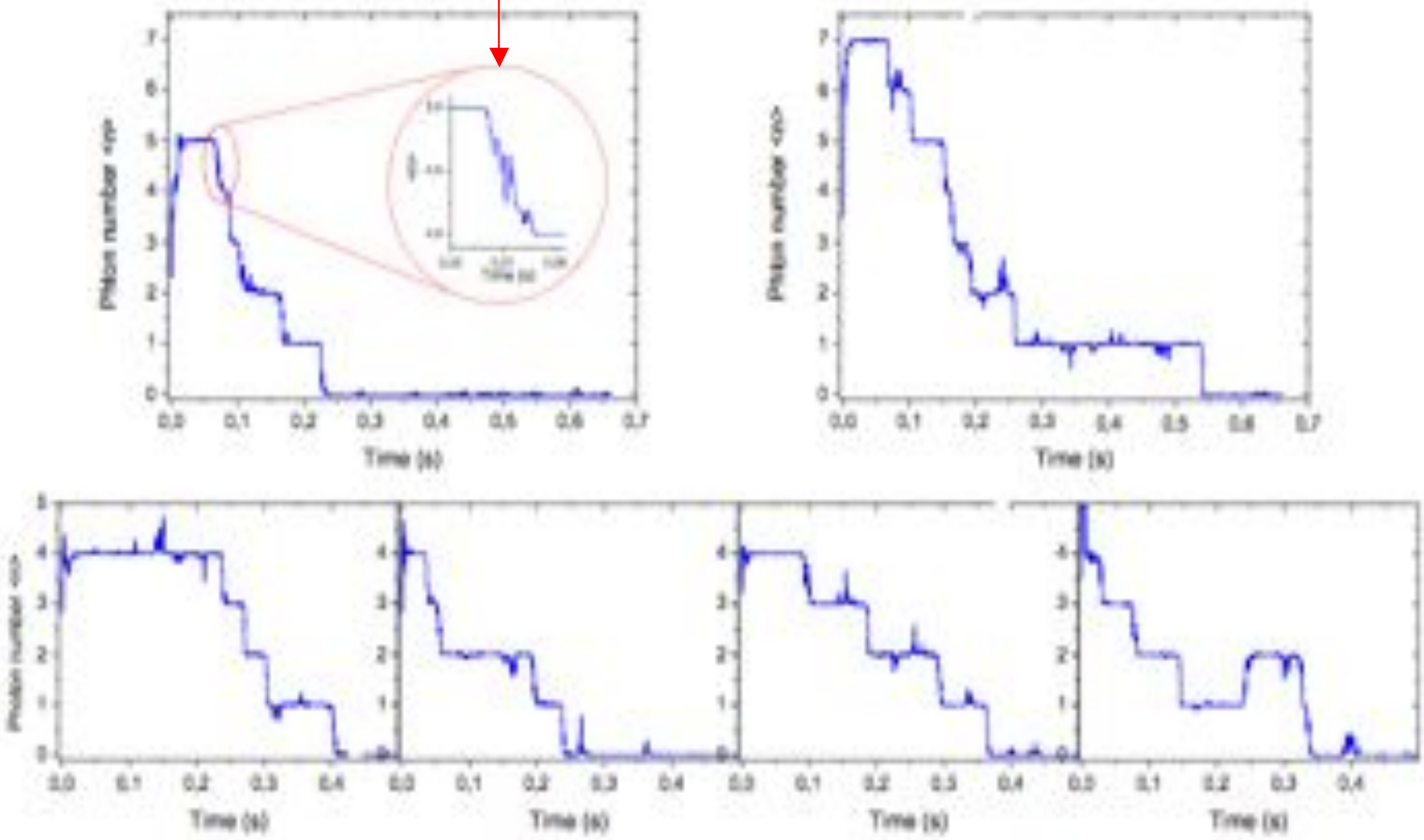
From probability $P_i(n)$ inferred after each atom, we deduce mean photon number:

$$\langle n \rangle = \sum_n n P_i(n)$$

Observation of stochastice field trajectories, in good agreement with Monte-Carlo calculations.

Other trajectories

It takes some time for atoms to recognize that a jump has occured.

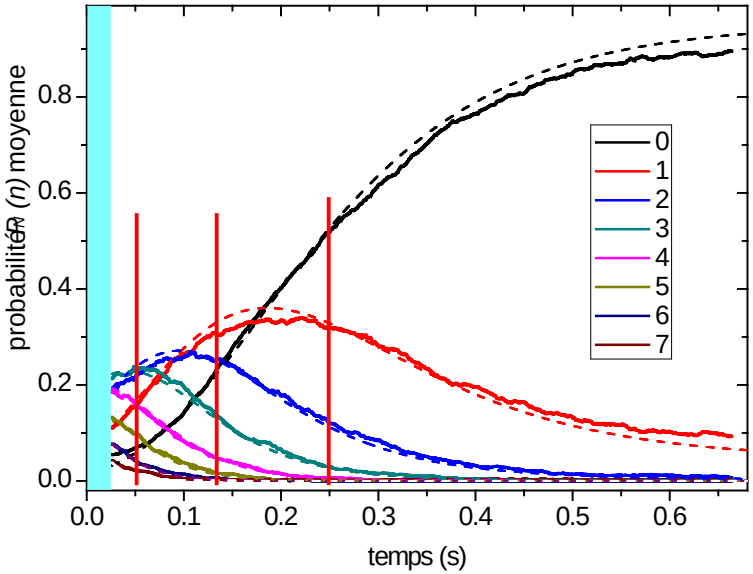


Four trajectories following a projection in n=4

A fundamentally random process (step durations fluctuate from one realization to the next one and only the statistics can be computed)

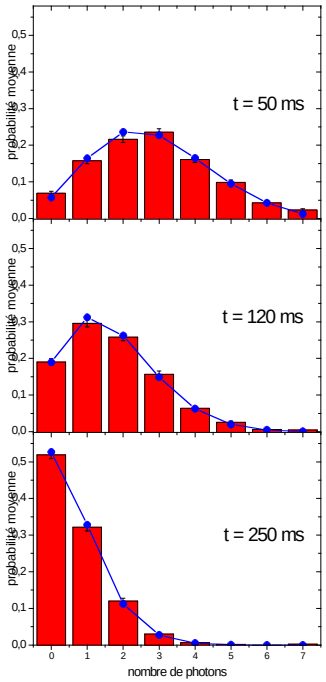
A statistical analysis of trajectories: field evolution versus time

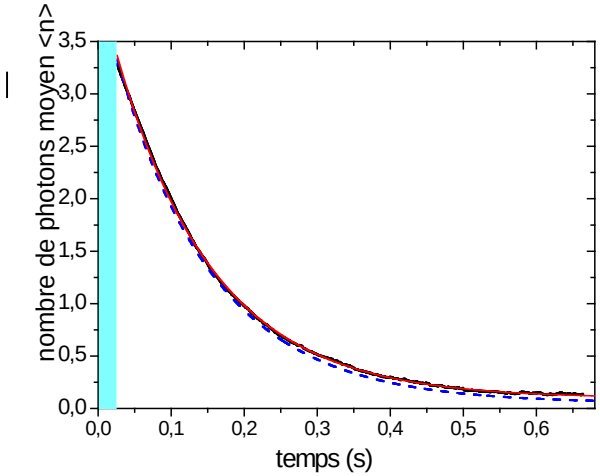
We analyse an *ensemble of trajectories* starting from the same initial coherent state and reconstruct $\Pi(n,t)$, the probability of finding n photons at time t (not to be confused with the probability $P_i^{(N)}(n)$ of the number of photons inferred after N atoms on one trajectory).



Left: $\Pi(n)$ versus time for an initial coherent state with n=3.5. Full lines: experiment, dotted lines: theory. The blue bar at t~0 indicates the dead time of the initial measurement.

Right: Histograms $\Pi(n)$ at times corresponding to the 3 vertical lines of the left f. Blue curves; theory. The photon distribution remains Poissonnian as expected for a damped coherent field.





At bottom: Evolution of average photon number

$$\bar{n}(t) = \sum_n n\Pi(n,t) \quad (6-11)$$

on ensemble of trajectories: the exponential law giving the damping of the field average energy is recovered.

Conclusion of second lecture

We have reviewed some general results about the quantum measurement theory and illustrated them by describing a photon counting experiment in cavity QED. We have shown that coupling a cavity field mode oscillator to a single two-level atom in the dispersive regime realizes a generalized measurement of the POVM kind which extracts partial information from the field. By a succession of such POVM's, a complete quantum non demolition (QND) measurement of the photon number is realized. This QND procedures allows us to observe directly the quantum jumps of the field and to prepare, by random projection, highly non-classical states. This experiment leads to the following questions:

- Can we reconstruct not only the photon number distribution but, more generally, the full quantum state of the field?
- The field convergence in a QND measurement is a random process. Can we interfere with this process by a feedback operation in order to drive the field on demand towards a given Fock state and what could be the use of such a procedure?
- Can all this be done in other systems?

I will discuss these issues in lectures 3 to 6.

General reference: S.Haroche and J-M.Raimond: *Exploring the Quantum: Atoms, Cavities and Photons*, Oxford University Press (2006).